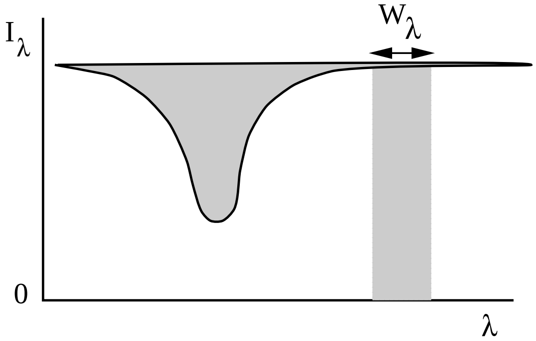
Chapter 1



Chapter 2

Radiation Quantities

Intensity

$$\begin{aligned} \mathrm{d}E_\nu &\equiv I_\nu(\vec{r},\vec{l},t)\,(\vec{l}\cdot\vec{n})\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega \qquad (2.1)\\ &= I_\nu(x,y,z,\theta,\varphi,t)\ \cos\theta\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega, \end{aligned}$$

Mean intensity

$$J_\nu(\vec{r},t) \equiv \frac{1}{4\pi}\int I_\nu\,\mathrm{d}\Omega = \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\pi} I_\nu \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi. \qquad (2.2)$$

Axial symmetry

$$J_\nu(z) = \frac{1}{4\pi}\int_0^{\pi} I_\nu(z,\theta)\,2\pi\sin\theta\,\mathrm{d}\theta = \frac{1}{2}\int_{-1}^{+1} I_\nu(z,\mu)\,\mathrm{d}\mu. \qquad (2.3)$$

Flux

$$\mathcal{F}_\nu(\vec{r},\vec{n},t) \equiv \int I_\nu \cos\theta\,\mathrm{d}\Omega = \int_0^{2\pi}\!\!\int_0^{\pi} I_\nu\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi. \qquad (2.4)$$

Incoming and outgoing

$$\begin{aligned} \mathcal{F}_\nu(z) &= \int_0^{2\pi}\!\!\int_0^{\pi/2} I_\nu\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi + \int_0^{2\pi}\!\!\int_{\pi/2}^{\pi} I_\nu\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\\ &= \int_0^{2\pi}\!\!\int_0^{\pi/2} I_\nu\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi - \int_0^{2\pi}\!\!\int_0^{\pi/2} I_\nu(\pi-\theta)\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\\ &\equiv \mathcal{F}_\nu^+(z) - \mathcal{F}_\nu^-(z), \qquad (2.5) \end{aligned}$$

Axial symmetry

$$\begin{aligned} \mathcal{F}_\nu(z) &= 2\pi\int_0^{\pi} I_\nu\cos\theta\sin\theta\,\mathrm{d}\theta\\ &= 2\pi\int_0^{1}\mu I_\nu\,\mathrm{d}\mu\ -\ 2\pi\int_0^{-1}\mu I_\nu\,\mathrm{d}\mu\\ &= \mathcal{F}_\nu^+(z)\ -\ \mathcal{F}_\nu^-(z). \qquad (2.6) \end{aligned}$$

Surface flux

$$\mathcal{F}_\nu^{\mathrm{surface}} \equiv \mathcal{F}_\nu^+(r\!=\!R) = \pi\overline{I_\nu^+}, \qquad (2.7)$$

Irradiance

$$\mathcal{R}_\nu = \frac{4\pi R^2}{4\pi D^2}\,\mathcal{F}_\nu^{\mathrm{surface}} = \frac{\pi R^2}{D^2}\,\overline{I}_\nu. \qquad (2.8)$$

Densities and Moments

Energy density

$$u_\nu = \frac{1}{c}\int I_\nu \, \mathrm{d}\Omega \tag{2.9}$$

LTE energy density

$$u = \int u_\nu \, \mathrm{d}\nu = \frac{1}{c}\iint B_\nu \, \mathrm{d}\Omega \, \mathrm{d}\nu = \frac{4\sigma}{c}T^4 \tag{2.10}$$

Photon density

$$N_{\rm photon} = \int_0^\infty \frac{u_\nu}{h\nu} \, \mathrm{d}\nu \approx 20\, T^3 \ \mathrm{cm}^{-3}. \tag{2.11}$$

Radiation pressure

$$p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta \, \mathrm{d}\Omega \tag{2.12}$$

Moments of the intensity

$$J_\nu(z) \equiv \frac{1}{2}\int_{-1}^{+1} I_\nu \, \mathrm{d}\mu \tag{2.13}$$

$$H_\nu(z) \equiv \frac{1}{2}\int_{-1}^{+1} \mu \, I_\nu \, \mathrm{d}\mu \tag{2.14}$$

$$K_\nu(z) \equiv \frac{1}{2}\int_{-1}^{+1} \mu^2 I_\nu \, \mathrm{d}\mu \tag{2.15}$$

Emission and Extinction

Emissivity

$$\mathrm{d}E_\nu \equiv j_\nu \,\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega, \tag{2.16}$$

$$\mathrm{d}I_\nu(s) = j_\nu(s)\,\mathrm{d}s, \tag{2.17}$$

Extinction coefficient

$$\mathrm{d}I_\nu \equiv -\sigma_\nu n\, I_\nu\,\mathrm{d}s \tag{2.18}$$

$$\mathrm{d}I_\nu \equiv -\alpha_\nu\, I_\nu\,\mathrm{d}s \tag{2.19}$$

$$\mathrm{d}I_\nu \equiv -\kappa_\nu \rho\, I_\nu\,\mathrm{d}s \tag{2.20}$$

Source function

$$S_\nu \equiv j_\nu/\alpha_\nu. \tag{2.21}$$

$$S_\nu^{\mathrm{tot}} = \frac{\sum j_\nu}{\sum \alpha_\nu}, \tag{2.22}$$

$$S_\nu^{\mathrm{tot}} = \frac{j_\nu^c + j_\nu^l}{\alpha_\nu^c + \alpha_\nu^l} = \frac{S_\nu^c + \eta_\nu S_\nu^l}{1 + \eta_\nu} \tag{2.23}$$

Transport equation

$$\mathrm{d}I_\nu(s) = I_\nu(s + \mathrm{d}s) - I_\nu(s) = j_\nu(s)\,\mathrm{d}s - \alpha_\nu(s) I_\nu(s)\,\mathrm{d}s \tag{2.24}$$

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu \tag{2.25}$$

$$\frac{\mathrm{d}I_\nu}{\alpha_\nu\,\mathrm{d}s} = S_\nu - I_\nu, \tag{2.26}$$

Simple Radiative Transfer

Optical path

$$\mathrm{d}\tau_\nu(s) \equiv \alpha_\nu(s)\,\mathrm{d}s; \tag{2.27}$$

Optical thickness

$$\tau_\nu(D) = \int_0^D \alpha_\nu(s)\,\mathrm{d}s, \tag{2.28}$$

Extinction only

$$I_\nu(D) = I_\nu(0)\,\mathrm{e}^{-\tau_\nu(D)}. \tag{2.29}$$

Mean free path

$$<\tau_\nu(s)> \equiv \frac{\int_0^\infty \tau_\nu(s)\,\mathrm{e}^{-\tau_\nu(s)}\,\mathrm{d}\tau_\nu(s)}{\int_0^\infty \mathrm{e}^{-\tau_\nu(s)}\,\mathrm{d}\tau_\nu(s)} = 1 \tag{2.30}$$

$$l_\nu = \frac{<\tau_\nu(s)>}{\alpha_\nu} = \frac{1}{\alpha_\nu} = \frac{1}{\kappa_\nu \rho}. \tag{2.31}$$

Transport equation

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = S_\nu - I_\nu, \tag{2.32}$$

Integral form

$$I_\nu(\tau_\nu) = I_\nu(0)\,\mathrm{e}^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(t_\nu)\,\mathrm{e}^{-(\tau_\nu - t_\nu)}\,\mathrm{d}t_\nu. \tag{2.33}$$

Homogeneous medium

$$I_\nu(D) = I_\nu(0)\,\mathrm{e}^{-\tau_\nu(D)} + S_\nu\left(1 - \,\mathrm{e}^{-\tau_\nu(D)}\right). \tag{2.34}$$

$\tau_\nu(D) \gg 1$

$$I_\nu(D) \approx S_\nu, \tag{2.35}$$

$\tau_\nu(D) \ll 1$

$$I_\nu(D) \approx I_\nu(0) + [S_\nu - I_\nu(0)]\;\tau_\nu(D). \tag{2.36}$$

Radiative Transfer in Stellar Atmospheres

Angle-dependent optical depth

$$\mathrm{d}\tau_{\nu\mu} \equiv -\alpha_\nu \frac{\mathrm{d}z}{|\mu|} \tag{2.37}$$

Radial optical depth

$$\tau_\nu(z_0) = \int_\infty^{z_0} -\alpha_\nu \,\mathrm{d}z = \int_{z_0}^\infty \alpha_\nu \,\mathrm{d}z, \tag{2.38}$$

Total optical depth

$$\mathrm{d}\tau_\nu^{\mathrm{total}} = -(\alpha_\nu^c + \alpha_\nu^l)\,\mathrm{d}z = (1+\eta_\nu)\,\mathrm{d}\tau_\nu^c \tag{2.39}$$

Standard transport equation

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu. \tag{2.40}$$

Formal solution

$$I_\nu^-(\tau_\nu,\mu) = -\int_0^{\tau_\nu} S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/\mu \tag{2.41}$$

$$I_\nu^+(\tau_\nu,\mu) = +\int_{\tau_\nu}^\infty S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/\mu. \tag{2.42}$$

Emergent intensity

$$I_\nu^+(\tau_\nu\!=\!0,\mu) = \int_0^\infty S_\nu(t_\nu)\,\mathrm{e}^{-t_\nu/\mu}\,\mathrm{d}t_\nu/\mu. \tag{2.43}$$

Eddington-Barbier approximation

$$I_\nu^+(\tau_\nu\!=\!0,\mu) \;\approx\; S_\nu(\tau_\nu=\mu) \tag{2.44}$$

$$\mathcal{F}_\nu^+(0) \approx \pi S_\nu(\tau_\nu = 2/3). \tag{2.45}$$

Bound-Bound Processes

Spontaneous deexcitation

$$A_{ul} \equiv \text{transition probability for spontaneous deexcitation from state } u \text{ to state } l \text{ per sec per particle in state } u. \tag{2.46}$$

Lorentz profile

$$\psi(\nu-\nu_0) = \frac{\gamma^{\mathrm{rad}}/4\pi^2}{(\nu-\nu_0)^2 + (\gamma^{\mathrm{rad}}/4\pi)^2}. \tag{2.47}$$

Voigt function

$$\psi(\nu-\nu_0) = \frac{H(a,v)}{\sqrt{\pi}\Delta\nu_{\mathrm{D}}} \tag{2.48}$$

Doppler width

$$\Delta\nu_{\mathrm{D}} \equiv \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}, \tag{2.49}$$

Radiative excitation

$$B_{lu}\overline{J}^{\varphi}_{\nu_0} \equiv \text{number of radiative excitations from state } l \text{ to state } u \text{ per sec per particle in state } l, \tag{2.50}$$

Averaged mean intensity

$$\overline{J}^{\varphi}_{\nu_0} \equiv \int_0^\infty J_\nu\, \varphi(\nu-\nu_0)\, \mathrm{d}\nu, \tag{2.51}$$

$$\overline{J}^{\varphi}_{\nu_0} \equiv \frac{1}{2}\int_0^\infty\int_{-1}^{+1} I_\nu\, \varphi(\nu-\nu_0)\, \mathrm{d}\mu\, \mathrm{d}\nu, \tag{2.52}$$

Extinction profile

$$\varphi(\nu-\nu_0) = \frac{H(a,v)}{\sqrt{\pi}\Delta\nu_{\mathrm{D}}}. \tag{2.53}$$

Amplitude

$$\varphi(\nu=\nu_0) = \frac{1-a}{\sqrt{\pi}\Delta\nu_{\mathrm{D}}}, \tag{2.54}$$

Induced deexcitation

$$B_{ul}\overline{J}^{\chi}_{\nu_0} \equiv \text{number of induced radiative deexcitations from state } u \text{ to state } l \text{ per sec per particle in state } u, \tag{2.55}$$

$$\overline{J}^{\chi}_{\nu_0} \equiv \frac{1}{2}\int_0^{\infty}\!\!\int_{-1}^{+1} I_\nu\,\chi(\nu-\nu_0)\,\mathrm{d}\mu\,\mathrm{d}\nu = \int_0^{\infty} J_\nu\chi(\nu-\nu_0)\,\mathrm{d}\nu \tag{2.56}$$

Collision probabilities

$$C_{lu} \equiv \text{number of collisional excitations from state } l \text{ to state } u \text{ per sec per particle in state } l. \tag{2.57}$$

$$C_{ul} \equiv \text{number of collisional deexcitations from state } u \text{ to state } l \text{ per sec per particle in state } u \tag{2.58}$$

$$n_i C_{ij} = n_i N_\mathrm{e}\int_{v_0}^{\infty}\sigma_{ij}(v)\,v\,f(v)\,\mathrm{d}v, \tag{2.59}$$

Einstein relations

$$\frac{B_{lu}}{B_{ul}} = \frac{g_u}{g_l} \qquad\qquad \frac{A_{ul}}{B_{ul}} = \frac{2h\nu^3}{c^2} \tag{2.60}$$

$$\frac{C_{ul}}{C_{lu}} = \frac{g_l}{g_u}\,\mathrm{e}^{E_{ul}/kT}, \tag{2.61}$$

Formal Coefficients

Monochromatic line extinction

$$\alpha^l_\nu = \frac{h\nu}{4\pi}\left[n_l B_{lu}\,\varphi(\nu-\nu_0) - n_u B_{ul}\,\chi(\nu-\nu_0)\right] \tag{2.62}$$

$$= \frac{h\nu}{4\pi}\,n_l B_{lu}\,\varphi(\nu-\nu_0)\left[1 - \frac{n_u\,g_l\,\chi(\nu-\nu_0)}{n_l\,g_u\,\varphi(\nu-\nu_0)}\right] \tag{2.63}$$

Total line extinction

$$\alpha^l_{\nu_0} \equiv \int_0^\infty \alpha^l_\nu\,\mathrm{d}\nu = \frac{h\nu_0}{4\pi}\,\left(n_l B_{lu} - n_u B_{ul}\right) \tag{2.64}$$

Per particle

$$\sigma^l_\nu = \frac{h\nu}{4\pi}\,B_{lu}\,\varphi(\nu-\nu_0). \tag{2.65}$$

Oscillator strength

$$\sigma^l_{\nu_0} \equiv \int_0^\infty \sigma^l_\nu\,\mathrm{d}\nu = \frac{h\nu_0}{4\pi}\,B_{lu} = \frac{\pi e^2}{m_\mathrm{e} c} f_{lu} = 0.02654\,f_{lu}\ \ \mathrm{cm}^2\,\mathrm{Hz}. \tag{2.66}$$

A_{ul} and f_{lu}

$$A_{ul} \sim \frac{g_l}{g_u}\,f_{lu}\,(\Delta E_{ul})^2 \tag{2.67}$$

Numerically

$$A_{ul} = 6.67 \times 10^{13}\frac{g_l}{g_u}\,\frac{f_{lu}}{\lambda^2}\ \ \mathrm{s}^{-1} \tag{2.68}$$

Monochromatic line emission

$$j^l_\nu = \frac{h\nu}{4\pi} n_u A_{ul}\,\psi(\nu-\nu_0). \tag{2.69}$$

Total line emission

$$j^l_{\nu_0} = \int_0^\infty j^l_\nu\,\mathrm{d}\nu = \frac{h\nu_0}{4\pi} n_u A_{ul} \tag{2.70}$$

<u>Formal Source Function</u>

General source function

$$S_\nu^l \equiv j_\nu^l/\alpha_\nu^l = \frac{n_u A_{ul}\psi(\nu-\nu_0)}{n_l B_{lu}\varphi(\nu-\nu_0) - n_u B_{ul}\chi(\nu-\nu_0)} \tag{2.71}$$

$$S_\nu^l = \frac{\dfrac{A_{ul}}{B_{ul}}\dfrac{\psi}{\varphi}}{\dfrac{n_l}{n_u}\dfrac{B_{lu}}{B_{ul}} - \dfrac{\chi}{\varphi}} = \frac{2h\nu^3}{c^2}\frac{\psi/\varphi}{\dfrac{g_u n_l}{g_l n_u} - \dfrac{\chi}{\varphi}}. \tag{2.72}$$

Complete redistribution

$$S_{\nu_0}^l = \frac{n_u A_{ul}}{n_l B_{lu} - n_u B_{ul}} = \frac{2h\nu_0^3}{c^2}\frac{1}{\dfrac{g_u n_l}{g_l n_u} - 1}. \tag{2.73}$$

Continuum Transitions

Bound-free extinction (Kramers)

$$\sigma_\nu^{\rm bf} = 2.815 \times 10^{29} \frac{Z^4}{n^5 \nu^3} g_{\rm bf} \qquad \text{for } \nu \geq \nu_0, \tag{2.74}$$

LTE

$$\alpha_\nu^{\rm bf} = \sigma_\nu^{\rm bf}\, n_i \left(1 - \mathrm{e}^{-h\nu/kT}\right) \tag{2.75}$$

Free-free extinction

$$\sigma_\nu^{\rm ff} = 3.7 \times 10^8\, N_{\rm e} \frac{Z^2}{T^{1/2}\nu^3}\, g_{\rm ff}, \tag{2.76}$$

Volume coefficient

$$\alpha_\nu^{\rm ff} = \sigma_\nu^{\rm ff}\, N_{\rm ion} \left(1 - \mathrm{e}^{-h\nu/kT}\right) \tag{2.77}$$

Wien

$$\alpha_\nu^{\rm ff} \approx 3.7 \times 10^8\, N_{\rm e} N_{\rm ion} \frac{Z^2}{T^{1/2}\nu^3}\, g_{\rm ff} \tag{2.78}$$

Rayleigh-Jeans

$$\alpha_\nu^{\rm ff} \approx 0.018\, N_{\rm e} N_{\rm ion} \frac{Z^2}{T^{3/2}\nu^2}\, g_{\rm ff} \tag{2.79}$$

Thomson scattering

$$\sigma_\nu^{\rm T} \equiv \sigma^{\rm T} = \frac{8\pi}{3} r_{\rm e}^2 = 6.65 \times 10^{-25}\ \mathrm{cm}^2. \tag{2.80}$$

$$\alpha_\nu^{\rm T} = \sigma^{\rm T}\, N_{\rm e} \tag{2.81}$$

Rayleigh scattering

$$\sigma_\nu^{\rm R} \approx f_{lu}\, \sigma^{\rm T} \left(\frac{\nu}{\nu_0}\right)^4, \tag{2.82}$$

$$\alpha_\nu^{\rm R} = \sigma_\nu^{\rm R}\, N_{\rm H}. \tag{2.83}$$

Matter in LTE

Maxwell distribution

$$\left[\frac{n(v_x)}{N}\,\mathrm{d}v_x\right]_{\mathrm{LTE}} = \left(\frac{m}{2\pi kT}\right)^{1/2}\,\mathrm{e}^{-(1/2)mv_x^2/kT}\,\mathrm{d}v_x, \tag{2.84}$$

$$\left[\frac{n(v)}{N}\,\mathrm{d}v\right]_{\mathrm{LTE}} = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2\,\mathrm{e}^{-(1/2)mv^2/kT}\,\mathrm{d}v. \tag{2.85}$$

Boltzmann distribution

$$\left[\frac{n_{r,s}}{n_{r,t}}\right]_{\mathrm{LTE}} = \frac{g_{r,s}}{g_{r,t}}\,\mathrm{e}^{-(\chi_{r,s}-\chi_{r,t})/kT}, \tag{2.86}$$

Saha distribution

$$\left[\frac{n_{r+1,1}}{n_{r,1}}\right]_{\mathrm{LTE}} = \frac{1}{N_{\mathrm{e}}}\,\frac{2\,g_{r+1,1}}{g_{r,1}}\left(\frac{2\pi m_{\mathrm{e}}kT}{h^2}\right)^{3/2}\,\mathrm{e}^{-\chi_r/kT}, \tag{2.87}$$

$$\left[\frac{N_{r+1}}{N_r}\right]_{\mathrm{LTE}} = \frac{1}{N_{\mathrm{e}}}\,\frac{2\,U_{r+1}}{U_r}\left(\frac{2\pi m_{\mathrm{e}}kT}{h^2}\right)^{3/2}\,\mathrm{e}^{-\chi_r/kT}, \tag{2.88}$$

Partition function

$$U_r \equiv \sum_s g_{r,s}\,\mathrm{e}^{-\chi_{r,s}/kT}. \tag{2.89}$$

Saha-Boltzmann

$$\left[\frac{n_c}{n_i}\right]_{\mathrm{LTE}} = \frac{1}{N_{\mathrm{e}}}\,\frac{2\,g_c}{g_i}\left(\frac{2\pi m_{\mathrm{e}}kT}{h^2}\right)^{3/2}\,\mathrm{e}^{-\chi_{ci}/kT} \tag{2.90}$$

Radiation in LTE

Source function

$$\left[S_\nu^l\right]_{\rm LTE} = \frac{2h\nu^3}{c^2}\frac{1}{\left[\dfrac{g_u n_l}{g_l n_u}\right]_{\rm LTE} - 1} \tag{2.91}$$

$$= \frac{2h\nu^3}{c^2}\frac{1}{{\rm e}^{h\nu/kT} - 1} \equiv B_\nu(T). \tag{2.92}$$

Wien

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2}\,{\rm e}^{-h\nu/kT}, \tag{2.93}$$

Rayleigh-Jeans

$$B_\nu(T) \approx \frac{2\nu^2 kT}{c^2}, \tag{2.94}$$

Stefan-Boltzmann

$$B(T) = \int_0^\infty B_\nu\,{\rm d}\nu = \frac{\sigma}{\pi}T^4, \tag{2.95}$$

$$\sigma = \frac{2\pi^5 k^4}{15 h^3 c^2} = 5.67\times 10^{-5}\ \ {\rm erg\,cm^{-2}\,K^{-4}\,s^{-1}}. \tag{2.96}$$

Correction for induced emission

$$\left[1 - \frac{n_u B_{ul}\chi(\nu-\nu_0)}{n_l B_{lu}\varphi(\nu-\nu_0)}\right]_{\rm LTE} = 1 - \ {\rm e}^{-h\nu_0/kT}. \tag{2.97}$$

Line extinction

$$\left[\alpha_\nu^l\right]_{\rm LTE} = \frac{\pi e^2}{m_{\rm e}c} n_l^{\rm LTE}\, f_{lu}\,\varphi(\nu-\nu_0)\left[1 - \ {\rm e}^{-h\nu_0/kT}\right] \tag{2.98}$$

LTE $\equiv$ Maxwell + Saha + Boltzmann

$$S_\nu^l(\vec{r}) = B_\nu\left[T(\vec{r})\right] \qquad I_\nu(\vec{r},\vec{l}) \neq B_\nu\left[T(\vec{r})\right] \qquad J_\nu(\vec{r}) \neq B_\nu\left[T(\vec{r})\right] \qquad \mathcal{F}_\nu(\vec{r}) \neq 0. \tag{2.99}$$

NLTE Radiative Transfer

SE rate equations

$$\frac{\mathrm{d}n_i(\vec{r})}{\mathrm{d}t} = \sum_{j\neq i}^{N} n_j(\vec{r}) P_{ji}(\vec{r}) - n_i(\vec{r}) \sum_{j\neq i}^{N} P_{ij}(\vec{r}) = 0, \tag{2.100}$$

Rates per particle

$$P_{ij} = R_{ij} + C_{ij}. \tag{2.101}$$

Bound-bound radiative

$$R_{ij} = A_{ij} + B_{ij}\overline{J}_{\nu_0}. \tag{2.102}$$

Radiative transfer

$$\mu \frac{\mathrm{d}I_\nu(\vec{r},\mu)}{\mathrm{d}\tau_\nu(\vec{r})} = -S_\nu(\vec{r}) + I_\nu(\vec{r},\mu) \tag{2.103}$$

Population Departure Coefficients

Definition

$$b_l = n_l / n_l^{\rm LTE} \qquad\qquad b_u = n_u / n_u^{\rm LTE} \tag{2.104}$$

Line source function

$$S_\nu^l = \frac{2h\nu^3}{c^2} \frac{\psi/\varphi}{\dfrac{b_l}{b_u}\,{\rm e}^{h\nu/kT} - \dfrac{\chi}{\varphi}} \tag{2.105}$$

Complete redistribution

$$S_{\nu_0}^l = \frac{2h\nu_0^3}{c^2} \frac{1}{\dfrac{b_l}{b_u}\,{\rm e}^{h\nu_0/kT} - 1}, \tag{2.106}$$

Wien

$$S_{\nu_0}^l \approx \frac{b_u}{b_l} B_{\nu_0}. \tag{2.107}$$

Monochromatic line extinction

$$\alpha_\nu^l = \frac{h\nu}{4\pi}\, b_l\, n_l^{\rm LTE} B_{lu} \varphi(\nu-\nu_0) \left[1 - \frac{b_u\, n_u^{\rm LTE}\, B_{ul}\, \chi}{b_l\, n_l^{\rm LTE}\, B_{lu}\, \varphi}\right] \tag{2.108}$$

$$= \frac{h\nu}{4\pi}\, b_l\, n_l^{\rm LTE} B_{lu} \varphi(\nu-\nu_0) \left[1 - \frac{b_u}{b_l}\frac{\chi}{\varphi}\,{\rm e}^{-h\nu/kT}\right] \tag{2.109}$$

$$= b_l\, n_l^{\rm LTE} \sigma_\nu^l \left[1 - \frac{b_u}{b_l}\frac{\chi}{\varphi}\,{\rm e}^{-h\nu/kT}\right] \tag{2.110}$$

$$= \frac{\pi e^2}{m_{\rm e} c}\, b_l\, n_l^{\rm LTE} f_{lu} \varphi(\nu-\nu_0) \left[1 - \frac{b_u}{b_l}\frac{\chi}{\varphi}\,{\rm e}^{-h\nu/kT}\right] \tag{2.111}$$

Wien

$$\alpha_\nu^l \approx b_l \left[\alpha_\nu^l\right]_{\rm LTE}. \tag{2.112}$$

Total line extinction

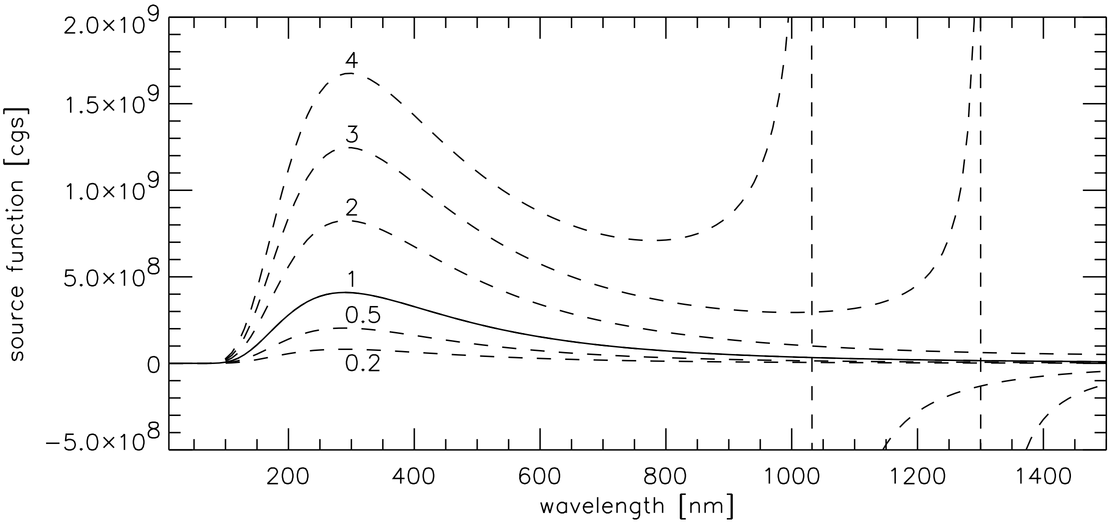
$$\alpha_{\nu_0}^l = \frac{h\nu_0}{4\pi}\, b_l\, n_l^{\rm LTE} B_{lu} \left[1 - \frac{b_u}{b_l}\,{\rm e}^{-h\nu_0/kT}\right] \tag{2.113}$$

$$= \frac{\pi e^2}{m_{\rm e} c}\, b_l\, n_l^{\rm LTE} f_{lu} \left[1 - \frac{b_u}{b_l}\,{\rm e}^{-h\nu_0/kT}\right] \tag{2.114}$$

$$\approx b_l \left[\alpha_{\nu_0}^l\right]_{\rm LTE}. \tag{2.115}$$

Reversed ratios

$$\frac{S_{\nu_0}^l}{B_{\nu_0}} = \frac{1 - {\rm e}^{-h\nu_0/kT}}{(b_l/b_u)\left[1 - (b_u/b_l)\,{\rm e}^{-h\nu_0/kT}\right]} = b_u \frac{\left[\alpha_{\nu_0}^l\right]_{\rm LTE}}{\alpha_{\nu_0}^l}, \tag{2.116}$$



Bound-free source function

$$S_\nu^{\rm bf} = \frac{2h\nu^3}{c^2} \frac{1}{\dfrac{b_i}{b_c}\,{\rm e}^{h\nu/kT} - 1} \tag{2.117}$$

Wien

$$S_\nu^{\rm bf} \approx \frac{b_c}{b_i} B_\nu. \tag{2.118}$$

Bound-free extinction coefficient

$$\alpha_\nu^{\rm bf} = b_i\, n_i^{\rm LTE} \sigma_{ic}(\nu) \left(1 - \frac{b_c}{b_i}\,{\rm e}^{-h\nu/kT}\right). \tag{2.119}$$

Bound-free emissivity

$$j_\nu^{\rm bf} = \alpha_\nu^{\rm bf}\, S_\nu^{\rm bf} = b_c \left[\alpha_\nu^{\rm bf}\right]_{\rm LTE} B_\nu \tag{2.120}$$

Free-free

$$S_\nu^{\rm ff} \;=\; B_\nu \tag{2.121}$$
$$\alpha_\nu^{\rm ff} \;=\; b_c\, n_c^{\rm LTE} \sigma_\nu^{\rm ff} \left(1 - \,{\rm e}^{-h\nu/kT}\right) \tag{2.122}$$
$$j_\nu^{\rm ff} \;=\; b_c \left[\alpha_\nu^{\rm ff}\right]_{\rm LTE} B_\nu \tag{2.123}$$

Departure Coefficient Conventions

Two conventions

$$b_i^{\rm Zwaan} \equiv n_i/n_i^{\rm LTE} \tag{2.124}$$

$$b_i^{\rm Menzel} \equiv \frac{n_i/n_i^{\rm LTE}}{n_C/n_C^{\rm LTE}} \tag{2.125}$$

Predominantly ionized

$$b_i^{\rm Menzel} \approx n_i/n_i^{\rm LTE} \approx b_i^{\rm Zwaan}, \tag{2.126}$$

Predominantly neutral

$$b_i^{\rm Menzel} \approx n_C^{\rm LTE}/n_C \approx 1/b_C^{\rm Zwaan}. \tag{2.127}$$

Formal Temperatures

Excitation temperature

$$\frac{n_u}{n_l} \equiv \frac{g_u}{g_l}\, \mathrm{e}^{-h\nu/kT_{\mathrm{exc}}} \tag{2.128}$$

Line source function

$$S^l_{\nu_0} = \frac{2h\nu_0^3}{c^2} \frac{1}{\dfrac{g_u n_l}{g_l n_u} - 1} = \frac{2h\nu_0^3}{c^2} \frac{1}{\mathrm{e}^{h\nu_0/kT_{\mathrm{exc}}} - 1} = B_{\nu_0}(T_{\mathrm{exc}}). \tag{2.129}$$

Bound-free source function

$$S^{\mathrm{bf}}_\nu \equiv \frac{2h\nu^3}{c^2} \frac{1}{\mathrm{e}^{h\nu/kT_{\mathrm{ion}}} - 1} = B_\nu(T_{\mathrm{ion}}). \tag{2.130}$$

Radiation temperature

$$B_\nu(T_{\mathrm{rad}}) \equiv J_\nu. \tag{2.131}$$

Brightness temperature

$$B_\nu(T_{\mathrm{b}}) \equiv I_\nu, \tag{2.132}$$

Effective temperature

$$\pi B(T_{\mathrm{eff}}) = \sigma T^4_{\mathrm{eff}} \equiv \mathcal{F}_{\mathrm{surface}} \tag{2.133}$$

Coherently Scattering Two-Level Atoms

Extinction

$$\alpha^l_\nu = \alpha^{\mathrm{a}}_\nu + \alpha^{\mathrm{s}}_\nu. \tag{2.134}$$

Destruction probability

$$\varepsilon_\nu \equiv \frac{\alpha^{\mathrm{a}}_\nu}{\alpha^{\mathrm{a}}_\nu + \alpha^{\mathrm{s}}_\nu}. \tag{2.135}$$

Scattering probability

$$1 - \varepsilon_\nu = \frac{\alpha^{\mathrm{s}}_\nu}{\alpha^{\mathrm{a}}_\nu + \alpha^{\mathrm{s}}_\nu}. \tag{2.136}$$

Effective path

$$l^*_\nu \approx \sqrt{N}\, l_\nu, \tag{2.137}$$

$$l_\nu = \frac{\langle \tau_\nu \rangle}{\alpha_\nu} = \frac{1}{\alpha^{\mathrm{a}}_\nu + \alpha^{\mathrm{s}}_\nu}. \tag{2.138}$$

$$l^*_\nu \approx l_\nu / \sqrt{\varepsilon_\nu} \tag{2.139}$$

Effective optical thickness

$$\tau^*_\nu = \sqrt{\varepsilon_\nu}\, \tau_\nu, \tag{2.140}$$

Effective radial optical depth

$$\mathrm{d}\tau^*_\nu = \sqrt{\varepsilon_\nu}\, \mathrm{d}\tau_\nu, \tag{2.141}$$

Coherent Two-Level Transport

Thermal absorption

$$j_\nu^{\mathrm{a}} = \alpha_\nu^{\mathrm{a}} B_\nu. \tag{2.142}$$

Elastic scattering

$$j_\nu^{\mathrm{s}} = \alpha_\nu^{\mathrm{s}} J_\nu. \tag{2.143}$$

Coherent scattering

$$S_\nu^l = \frac{j_\nu^{\mathrm{a}} + j_\nu^{\mathrm{s}}}{\alpha_\nu^{\mathrm{a}} + \alpha_\nu^{\mathrm{s}}} = (1 - \varepsilon_\nu)\, J_\nu + \varepsilon_\nu B_\nu. \tag{2.144}$$

Complete redistribution

$$S_{\nu_0}^l = (1 - \varepsilon_{\nu_0})\, \overline{J}_{\nu_0}^{\varphi} + \varepsilon_{\nu_0} B_{\nu_0}, \tag{2.145}$$

Profile-summed destruction probability

$$\varepsilon_{\nu_0} \equiv \frac{\alpha_{\nu_0}^{\mathrm{a}}}{\alpha_{\nu_0}^{\mathrm{a}} + \alpha_{\nu_0}^{\mathrm{s}}}. \tag{2.146}$$

Two-level transport

$$\mathrm{d}I_\nu = -\alpha_\nu^{\mathrm{a}} I_\nu \,\mathrm{d}s - \alpha_\nu^{\mathrm{s}} I_\nu \,\mathrm{d}s + \alpha_\nu^{\mathrm{a}} B_\nu \,\mathrm{d}s + \alpha_\nu^{\mathrm{s}} J_\nu \,\mathrm{d}s \tag{2.147}$$

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = \frac{\mathrm{d}I_\nu}{(\alpha_\nu^{\mathrm{a}} + \alpha_\nu^{\mathrm{s}})\,\mathrm{d}s} = S_\nu^l - I_\nu \tag{2.148}$$

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu^l. \tag{2.149}$$

Chapter 3

Bound-Bound Rates

SE rate equations

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = \sum_{j\neq i}^{N} n_j P_{ji} - n_i \sum_{j\neq i}^{N} P_{ij} = 0 \tag{3.1}$$

Rates per particle

$$P_{ij} = A_{ij} + B_{ij}\overline{J}_{\nu_0} + C_{ij}. \tag{3.2}$$

Radiative excitation rate

$$\begin{aligned} n_l R_{lu} &= n_l B_{lu} \overline{J}_{\nu_0} && (3.3) \\ &= n_l \int_0^\infty B_{lu}\, J_\nu\, \varphi(\nu-\nu_0)\, \mathrm{d}\nu \\ &= 4\pi\, n_l \int_0^\infty \frac{\sigma_\nu^l}{h\nu}\, J_\nu\, \mathrm{d}\nu && (3.4) \end{aligned}$$

Radiative deexcitation rate

$$\begin{aligned} n_u R_{ul} &= n_u A_{ul} + n_u B_{ul} \overline{J}_{\nu_0} && (3.5) \\ &= n_u \int_0^\infty A_{ul}\, \varphi(\nu-\nu_0)\, \mathrm{d}\nu + n_u \int_0^\infty B_{ul}\, J_\nu\, \varphi(\nu-\nu_0)\, \mathrm{d}\nu \\ &= n_u \frac{g_l}{g_u} \int_0^\infty B_{lu}\, \varphi(\nu-\nu_0) \left(\frac{2h\nu^3}{c^2} + J_\nu\right) \mathrm{d}\nu \\ &= 4\pi\, n_u \frac{g_l}{g_u} \int_0^\infty \frac{\sigma_\nu^l}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_\nu\right) \mathrm{d}\nu. && (3.6) \end{aligned}$$

Bound-Free Rates

Photoionization rate

$$n_i R_{ic} = 4\pi\, n_i \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} J_\nu \,\mathrm{d}\nu \tag{3.7}$$

TE recombination rate

$$[n_c R_{ci}]_{\mathrm{TE}} = [n_i R_{ic}]_{\mathrm{TE}} = 4\pi\, n_i^{\mathrm{TE}} \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} B_\nu \,\mathrm{d}\nu. \tag{3.8}$$

Split spontaneous – induced

$$\begin{aligned} [n_c R_{ci}]_{\mathrm{TE}} &= 4\pi\, n_i^{\mathrm{TE}} \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} B_\nu \left[\left(1 - \mathrm{e}^{-h\nu/kT}\right) + \mathrm{e}^{-h\nu/kT}\right] \mathrm{d}\nu, && (3.9)\\ &= \left[n_c R_{ci}^{\mathrm{spon}}\right]_{\mathrm{TE}} + \left[n_c R_{ci}^{\mathrm{ind}}\right]_{\mathrm{TE}} && (3.10)\end{aligned}$$

TE spontaneous recombination rate

$$\left[n_c R_{ci}^{\mathrm{spon}}\right]_{\mathrm{TE}} = 4\pi\, n_i^{\mathrm{TE}} \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} B_\nu \left(1 - \mathrm{e}^{-h\nu/kT}\right) \mathrm{d}\nu. \tag{3.11}$$

Actual spontaneous recombination rate

$$\begin{aligned} n_c R_{ci}^{\mathrm{spon}} &= 4\pi\, \frac{n_c}{n_c^{\mathrm{LTE}}} n_i^{\mathrm{LTE}} \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} B_\nu \left(1 - \mathrm{e}^{-h\nu/kT}\right) \mathrm{d}\nu && (3.12)\\ &= 4\pi\, n_c \left[\frac{n_i}{n_c}\right]_{\mathrm{LTE}} \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} \frac{2h\nu^3}{c^2}\, \mathrm{e}^{-h\nu/kT} \,\mathrm{d}\nu. && (3.13)\end{aligned}$$

TE induced recombination rate

$$\left[n_c R_{ci}^{\mathrm{ind}}\right]_{\mathrm{TE}} = 4\pi\, n_i^{\mathrm{TE}} \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} B_\nu\, \mathrm{e}^{-h\nu/kT} \,\mathrm{d}\nu, \tag{3.14}$$

Actual induced recombination rate

$$n_c R_{ci}^{\mathrm{ind}} = 4\pi\, n_c \left[\frac{n_i}{n_c}\right]_{\mathrm{LTE}} \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} J_\nu\, \mathrm{e}^{-h\nu/kT} \,\mathrm{d}\nu. \tag{3.15}$$

Total radiative recombination rate

$$n_c R_{ci} = 4\pi\, n_c \left[\frac{n_i}{n_c}\right]_{\mathrm{LTE}} \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_\nu\right) \mathrm{e}^{-h\nu/kT} \,\mathrm{d}\nu, \tag{3.16}$$

Unified rates

Bound-bound + bound-free

$$\text{upward} \quad i \rightarrow j: \quad R_{ij} = \frac{1}{2}\int_{-1}^{1}\int_{0}^{\infty} \frac{4\pi}{h\nu}\sigma_{ij} I_{\nu\mu}\, \mathrm{d}\nu\, \mathrm{d}\mu \tag{3.17}$$

$$= \int_{0}^{\infty} \frac{4\pi}{h\nu}\sigma_{ij} J_{\nu}\, \mathrm{d}\nu \tag{3.18}$$

$$\text{downward} \quad j \rightarrow i: \quad R_{ji} = \frac{1}{2}\int_{-1}^{1}\int_{0}^{\infty} \frac{4\pi}{h\nu}\sigma_{ij} G_{ij} \left(\frac{2h\nu^3}{c^2} + I_{\nu\mu}\right) \mathrm{d}\nu\, \mathrm{d}\mu \tag{3.19}$$

$$= \int_{0}^{\infty} \frac{4\pi}{h\nu}\sigma_{ij} G_{ij} \left(\frac{2h\nu^3}{c^2} + J_{\nu}\right) \mathrm{d}\nu \tag{3.20}$$

Bound-bound

$$\sigma_{ij} = \sigma_{\nu}^{l} = \frac{h\nu_{ij}}{4\pi} B_{ij}\, \varphi_{\nu\mu} \qquad G_{ij} = \frac{g_i}{g_j} = \left[\frac{n_i}{n_j}\right]_{\mathrm{LTE}} \mathrm{e}^{-h\nu/kT} \tag{3.21}$$

Bound-free

$$\sigma_{ij} = \sigma_{ic}(\nu) \qquad G_{ij} = \left[\frac{n_i}{n_c}\right]_{\mathrm{LTE}} \mathrm{e}^{-h\nu/kT}. \tag{3.22}$$

Net Radiative Rates

Net radiative recombination rate

$$
\begin{aligned}
& n_c R_{ci} - n_i R_{ic} \\
&= 4\pi \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} \left[n_i^{\mathrm{LTE}} b_c B_\nu \left(1 - \mathrm{e}^{-h\nu/kT}\right) + n_i^{\mathrm{LTE}} b_c J_\nu \, \mathrm{e}^{-h\nu/kT} - n_i J_\nu \right] \, \mathrm{d}\nu \\
&= 4\pi \, n_i^{\mathrm{LTE}} b_c \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} \left[B_\nu \left(1 - \mathrm{e}^{-h\nu/kT}\right) - \frac{b_i}{b_c} J_\nu \left(1 - \frac{b_c}{b_i} \, \mathrm{e}^{-h\nu/kT}\right) \right] \, \mathrm{d}\nu. \qquad (3.23)
\end{aligned}
$$

Wien limit

$$n_c R_{ci} - n_i R_{ic} = 4\pi \, n_i^{\mathrm{LTE}} b_c \int_{\nu_0}^{\infty} \frac{\sigma_{ic}(\nu)}{h\nu} \left(B_\nu - \frac{b_i}{b_c} J_\nu \right) \, \mathrm{d}\nu. \qquad (3.24)$$

Net radiative deexcitation rate

$$
\begin{aligned}
& n_u R_{ul} - n_l R_{lu} \\
&= n_u A_{ul} + n_u B_{ul} \overline{J}_{\nu_0} - n_l B_{lu} \overline{J}_{\nu_0} \\
&= S^l_{\nu_0} (n_l B_{lu} - n_u B_{ul}) - \overline{J}_{\nu_0} \, (n_l B_{lu} - n_u B_{ul}) \\
&= n_l B_{lu} \left[S^l_{\nu_0} \left(1 - \frac{n_u B_{ul}}{n_l B_{lu}}\right) - \overline{J}_{\nu_0} \left(1 - \frac{n_u B_{ul}}{n_l B_{lu}}\right) \right] \\
&= \frac{4\pi}{h\nu_0} \, n_l \, \sigma^l_{\nu_0} \left[S^l_{\nu_0} \left(1 - \frac{b_u}{b_l} \, \mathrm{e}^{-h\nu_0/kT}\right) - \overline{J}_{\nu_0} \left(1 - \frac{b_u}{b_l} \, \mathrm{e}^{-h\nu_0/kT}\right) \right] \qquad (3.25) \\
&= \frac{4\pi}{h\nu_0} \, n_l^{\mathrm{LTE}} \, b_u \, \sigma^l_{\nu_0} \left[B_{\nu_0} \left(1 - \mathrm{e}^{-h\nu_0/kT}\right) - \frac{b_l}{b_u} \overline{J}_{\nu_0} \left(1 - \frac{b_u}{b_l} \, \mathrm{e}^{-h\nu_0/kT}\right) \right] \qquad (3.26)
\end{aligned}
$$

Wien limit

$$n_u R_{ul} - n_l R_{lu} \approx \frac{4\pi}{h\nu_0} \, n_l^{\mathrm{LTE}} \, b_u \, \sigma^l_{\nu_0} \left(B_{\nu_0} - \frac{b_l}{b_u} \overline{J}_{\nu_0} \right). \qquad (3.27)$$

Faster derivation

$$
\begin{aligned}
n_u R_{ul} - n_l R_{lu} &= n_u A_{ul} + n_u B_{ul} \overline{J}_{\nu_0} - n_l B_{lu} \overline{J}_{\nu_0} \\
&= \frac{4\pi}{h\nu_0} \left(j^l_{\nu_0} - \alpha^l_{\nu_0} \overline{J}_{\nu_0} \right) \qquad (3.28) \\
&= \frac{4\pi}{h\nu_0} \alpha^l_{\nu_0} \left(S^l_{\nu_0} - \overline{J}_{\nu_0} \right) \qquad (3.29) \\
&\approx \frac{4\pi}{h\nu_0} \, b_u \, \left[\alpha^l_{\nu_0} \right]_{\mathrm{LTE}} \left(B_{\nu_0} - \frac{b_l}{b_u} \overline{J}_{\nu_0} \right) \qquad (3.30)
\end{aligned}
$$

Collision Rates

Collision frequency

$$\frac{\text{electron collision frequency}}{\text{ion collision frequency}} \sim \frac{N_\mathrm{e} <v_\mathrm{e}>}{N_\mathrm{ion} <v_\mathrm{ion}>} \sim \frac{N_\mathrm{e}}{N_\mathrm{ion}} \left(\frac{m_\mathrm{H} A}{m_\mathrm{e}}\right)^{1/2} \tag{3.31}$$

Bound-bound for atoms

$$n_l C_{lu} \approx 2.16 \left(\frac{E_0}{kT}\right)^{-1.68} T^{-3/2}\, \mathrm{e}^{-E_0/kT}\, n_l\, N_\mathrm{e}\, f \tag{3.32}$$

$$n_u C_{ul} \approx 2.16 \left(\frac{E_0}{kT}\right)^{-1.68} T^{-3/2} \frac{g_l}{g_u}\, n_u\, N_\mathrm{e}\, f \tag{3.33}$$

Bound-bound for ions

$$n_l C_{lu} \approx 3.9 \left(\frac{E_0}{kT}\right)^{-1} T^{-3/2}\, \mathrm{e}^{-E_0/kT}\, n_l\, N_\mathrm{e}\, f \tag{3.34}$$

$$n_u C_{ul} \approx 3.9 \left(\frac{E_0}{kT}\right)^{-1} T^{-3/2} \frac{g_l}{g_u}\, n_u N_\mathrm{e}\, f \tag{3.35}$$

Bound-free

$$n_i C_{ic} \approx 2.7\,\zeta \left(\frac{E_0}{kT}\right)^{-2} T^{-3/2}\, \mathrm{e}^{-E_0/kT}\, n_i\, N_\mathrm{e} \tag{3.36}$$

$$n_c C_{ci} \approx 5.6 \times 10^{-16}\,\zeta \left(\frac{E_0}{kT}\right)^{-2} T^{-3}\, \frac{g_i}{g_c}\, n_c\, N_\mathrm{e}^2 \tag{3.37}$$

TE Saha ratio

$$\frac{C_{ci}}{C_{ic}} = \left[\frac{n_i}{n_c}\right]_\mathrm{LTE} = 2.06 \times 10^{-16}\, (g_i/g_c)\ \mathrm{e}^{+E_0/kT}\, T^{-3/2}\, N_\mathrm{e} \tag{3.38}$$

Ratios

$$\frac{n_u C_{ul}}{n_l C_{lu}} = \frac{b_u}{b_l} \tag{3.39}$$

$$\frac{n_c C_{ci}}{n_i C_{ic}} = \frac{b_c}{b_i}. \tag{3.40}$$

Net rates

$$n_u C_{ul} - n_l C_{lu} = n_l C_{lu} \left(\frac{b_u}{b_l} - 1\right) = b_u n_l^\mathrm{LTE} C_{lu} \left(1 - \frac{b_l}{b_u}\right) \tag{3.41}$$

$$n_c C_{ci} - n_i C_{ic} = n_i C_{ic} \left(\frac{b_c}{b_i} - 1\right) = b_c n_i^\mathrm{LTE} C_{ic} \left(1 - \frac{b_i}{b_c}\right), \tag{3.42}$$

Radiation and Collision Damping

Two-level atom

$$\gamma^{\mathrm{rad}} = \gamma_u^{\mathrm{rad}} = A_{ul} \tag{3.43}$$

Lorentz profile

$$\psi(\nu - \nu_0) = \frac{\gamma^{\mathrm{rad}}/4\pi^2}{(\nu-\nu_0)^2 + (\gamma^{\mathrm{rad}}/4\pi)^2} \tag{3.44}$$

Damped harmonic oscillator

$$\gamma^{\mathrm{rad}} = \frac{8\pi e^2}{3m_{\mathrm{e}}c\lambda_0^2} \tag{3.45}$$

Total damping per upper level

$$\gamma_u^{\mathrm{rad}} = \sum_{l<u} A_{ul} \tag{3.46}$$

Both levels

$$\gamma^{\mathrm{rad}} = \gamma_l^{\mathrm{rad}} + \gamma_u^{\mathrm{rad}} = \sum_{i<l} A_{li} + \sum_{i<u} A_{ui}, \tag{3.47}$$

Collisional interactions

$$\Delta\nu = \frac{\Delta E}{h} \equiv \frac{C_n}{r^n} \tag{3.48}$$

Van der Waals broadening

$$\log\gamma_6 \approx 6.33 + 0.4\log(\overline{r_u^2} - \overline{r_l^2}) + \log P_g - 0.7\log T \tag{3.49}$$

Bates-Damgaard approximation

$$\overline{r^2} = \frac{n^{*2}}{2Z^2}\left(5n^{*2} + 1 - 3l(l+1)\right). \tag{3.50}$$

Effective principal quantum number

$$n^{*2} = R\frac{Z^2}{E_\infty - E_n} \tag{3.51}$$

Doppler broadening

Doppler shift

$$\frac{\Delta\nu}{\nu} = -\frac{\Delta\lambda}{\lambda} = \frac{\xi}{c} \tag{3.52}$$

Emission blueshift

$$\nu = \nu'(1+\xi/c) \approx \nu' + \nu_0\xi/c \tag{3.53}$$

Extinction redshift

$$\nu' = \nu(1-\xi/c) \approx \nu - \nu_0\xi/c. \tag{3.54}$$

Thermal motions

$$\frac{n(\xi)}{N}\,\mathrm{d}\xi = \frac{1}{\xi_0\sqrt{\pi}}\,\mathrm{e}^{-\xi^2/\xi_0^2}\,\mathrm{d}\xi \tag{3.55}$$

$$\xi_0 = \sqrt{\frac{2kT}{m}} = 12.85\sqrt{\frac{T}{10^4 A}} \tag{3.56}$$

Root-mean-square velocity component

$$<\xi^2>^{1/2} = \sqrt{\frac{kT}{m}} = \frac{\xi_0}{\sqrt{2}}. \tag{3.57}$$

Total line extinction

$$\sigma^l_{\nu_0} \equiv \int_0^\infty \sigma^l(\nu-\nu_0)\,\mathrm{d}\nu = \int_0^\infty \frac{h\nu}{4\pi}B_{lu}\varphi(\nu-\nu_0)\,\mathrm{d}\nu = \frac{\pi e^2}{m_\mathrm{e}c}f\int_0^\infty \varphi(\nu-\nu_0)\,\mathrm{d}\nu = \frac{\pi e^2}{m_\mathrm{e}c}f \tag{3.58}$$

Line extinction in the frame of the atom

$$\sigma^l_{\nu'} = \sigma^l(\nu'-\nu_0) = \sigma^l(\nu-\xi\nu_0/c-\nu_0). \tag{3.59}$$

Convolution

$$\sigma^l(\nu-\nu_0) = \frac{\int \sigma^l(\nu-\xi\nu_0/c-\nu_0)\,n(\xi)\,\mathrm{d}\xi}{\int n(\xi)\,\mathrm{d}\xi} = \int_{-\infty}^{+\infty} \sigma^l(\nu-\xi\nu_0/c-\nu_0)\,\frac{n(\xi)}{N}\,\mathrm{d}\xi \tag{3.60}$$

Thermal broadening only

$$\sigma^l(\nu-\nu_0) = \frac{\pi e^2}{m_\mathrm{e}c}f\int_{-\infty}^{+\infty} \delta(\nu-\xi\nu_0/c-\nu_0)\,\frac{n(\xi)}{N}\,\mathrm{d}\xi \tag{3.61}$$

Line extinction for pure thermal broadening

$$\sigma_\nu^l = \frac{\pi e^2}{m_\mathrm{e} c} f \, \frac{n\left[(\nu - \nu_0)\, c/\nu_0\right]}{N} = \frac{\sqrt{\pi} e^2}{m_\mathrm{e} c} \frac{f}{\Delta\nu_\mathrm{D}} \, \mathrm{e}^{-(\Delta\nu/\Delta\nu_\mathrm{D})^2} \tag{3.62}$$

Doppler width

$$\Delta\nu_\mathrm{D} \equiv \frac{\xi_0}{c}\nu_0 = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}} \tag{3.63}$$

Gaussian profile

$$\varphi(\nu - \nu_0) = \frac{1}{\sqrt{\pi}\,\Delta\nu_\mathrm{D}} \, \mathrm{e}^{-(\Delta\nu/\Delta\nu_\mathrm{D})^2}. \tag{3.64}$$

Wavelength units

$$\sigma_\lambda^l = \frac{\sqrt{\pi} e^2}{m_\mathrm{e} c} \frac{\lambda^2}{c} \frac{f}{\Delta\lambda_\mathrm{D}} \, \mathrm{e}^{-(\Delta\lambda/\Delta\lambda_\mathrm{D})^2} \tag{3.65}$$

$$\Delta\lambda_\mathrm{D} \equiv \frac{\xi_0}{c}\lambda_0 = \frac{\lambda_0}{c}\sqrt{\frac{2kT}{m}} \tag{3.66}$$

Thermal plus collisional broadening

$$\begin{aligned}
\sigma_\nu^l &= \left[\frac{\sqrt{\pi} e^2}{m_\mathrm{e} c} \frac{f}{\Delta\nu_\mathrm{D}} \, \mathrm{e}^{-(\Delta\nu/\Delta\nu_\mathrm{D})^2}\right] * \left[\frac{\gamma/4\pi^2}{(\nu' - \nu_0)^2 + (\gamma/4\pi)^2}\right] \\
&= \frac{\sqrt{\pi} e^2}{m_\mathrm{e} c} \frac{f}{\Delta\nu_\mathrm{D}} \int_{-\infty}^{+\infty} \frac{(\gamma/4\pi^2)\, \mathrm{e}^{-(\Delta\nu/\Delta\nu_\mathrm{D})^2}}{(\nu' - \nu_0)^2 + (\gamma/4\pi)^2} \, \mathrm{d}\nu \\
&= \frac{\sqrt{\pi} e^2}{m_\mathrm{e} c} \, \frac{f}{\Delta\nu_\mathrm{D}} H(a, v)
\end{aligned} \tag{3.67}$$

Voigt function

Definition

$$H(a,v) \;\equiv\; \frac{a}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-y^2}}{(v-y)^2+a^2}\,\mathrm{d}y \tag{3.68}$$

$$y \;\equiv\; \frac{\xi}{\xi_0}=\frac{\xi}{c}\frac{\nu_0}{\Delta\nu_\mathrm{D}}=\frac{\xi}{c}\frac{\lambda_0}{\Delta\lambda_\mathrm{D}} \tag{3.69}$$

$$v \;\equiv\; \frac{\nu-\nu_0}{\Delta\nu_\mathrm{D}}=\frac{\lambda-\lambda_0}{\Delta\lambda_\mathrm{D}} \tag{3.70}$$

$$a \;\equiv\; \frac{\gamma}{4\pi\Delta\nu_\mathrm{D}}=\frac{\lambda^2}{4\pi c}\frac{\gamma}{\Delta\lambda_\mathrm{D}}. \tag{3.71}$$

Extinction profile

$$\varphi(\nu-\nu_0)=\frac{H(a,v)}{\sqrt{\pi}\,\Delta\nu_\mathrm{D}}. \tag{3.72}$$

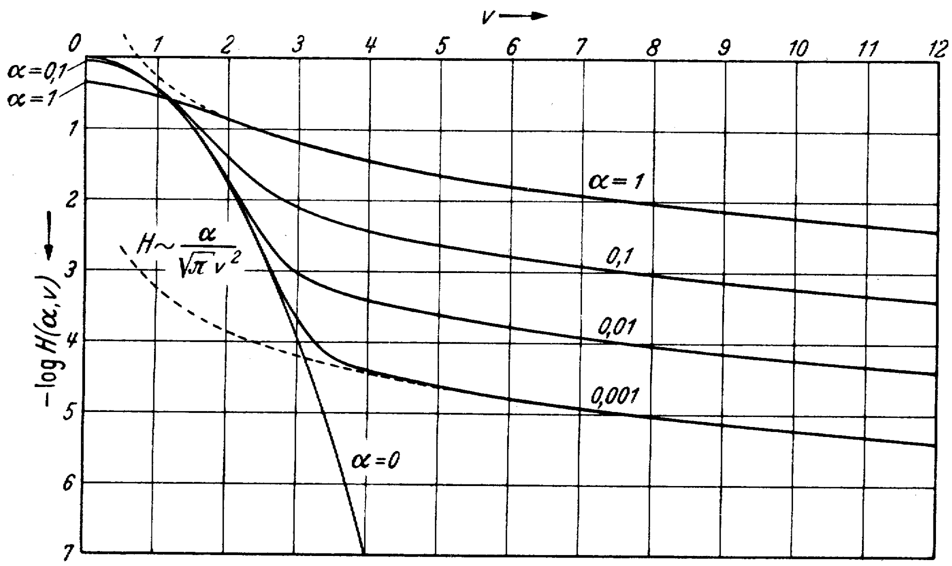
Approximation

$$H(a,v)\approx\;\mathrm{e}^{-v^2}+\frac{a}{\sqrt{\pi}\,v^2} \tag{3.73}$$

Far wings

$$\varphi(\nu-\nu_0)\approx\frac{1}{\sqrt{\pi}\Delta\nu_\mathrm{D}}\frac{a}{\sqrt{\pi}v^2}=\frac{1}{\sqrt{\pi}\Delta\nu_\mathrm{D}}\frac{\gamma}{4\pi\Delta\nu_\mathrm{D}}\frac{\Delta\nu_\mathrm{D}^2}{\sqrt{\pi}\Delta\nu^2}=\frac{\gamma}{4\pi^2\Delta\nu^2}. \tag{3.74}$$

Unsöld (1955)



Rotation and Turbulent Broadening

Line-of-sight component per x

$$v_z(x) = x\,\Omega\,\sin i, \tag{3.75}$$

Limb shift

$$\Delta\lambda_R = (\lambda/c) R\,\Omega\,\sin i = (\lambda/c)\,v\,\sin i \tag{3.76}$$

Flux spectrum

$$\mathcal{F}_\nu = \int I_\nu \cos\theta\,\mathrm{d}\Omega = \iint \frac{I_\nu(x,y)}{R^2}\,\mathrm{d}x\,\mathrm{d}y \tag{3.77}$$

Intensity profile

$$H(\nu) \equiv I(\nu)/I_\nu^{\rm cont} \tag{3.78}$$

Invariant intensity profile

$$\begin{aligned}
\frac{\mathcal{F}_\nu}{\mathcal{F}_\nu^{\rm cont}} &= \frac{\int H(\nu - \Delta\nu) I_\nu^{\rm cont} \cos\theta\,\mathrm{d}\Omega}{\int I_\nu^{\rm cont} \cos\theta\,\mathrm{d}\Omega} \\
&= \frac{\iint (H(\nu - \Delta\nu) I_\nu^{\rm cont}/R^2)\,\mathrm{d}x\,\mathrm{d}y}{\iint (I_\nu^{\rm cont}/R^2)\,\mathrm{d}x\,\mathrm{d}y} \\
&= \int_{-\infty}^{+\infty} H(\nu - \Delta\nu)\,G(\Delta\nu)\,\mathrm{d}\nu \\
&= H(\nu) * G(\nu)
\end{aligned} \tag{3.79}$$

Microturbulent broadening

$$\Delta\nu_{\rm D} \equiv \frac{\nu_0}{c}\sqrt{\frac{2kT}{m} + \xi_{\rm micro}^2} \tag{3.80}$$

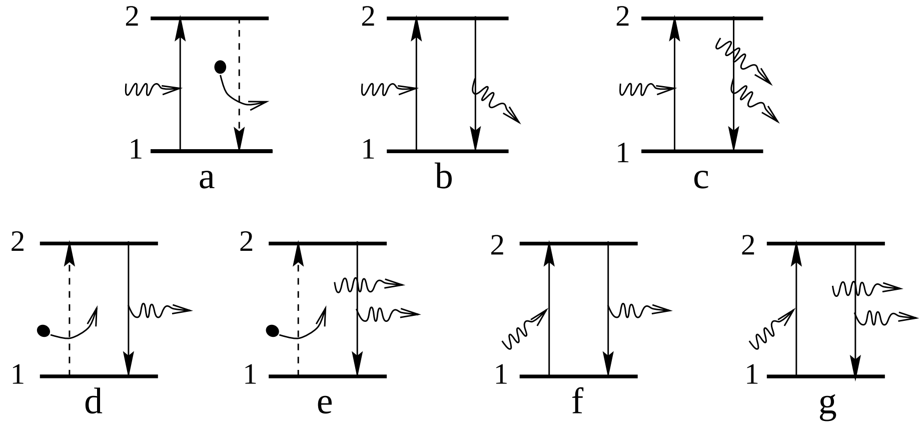
Macroturbulent broadening

$$\frac{I_{\rm c} - I_\lambda}{I_{\rm c}} = \left[\frac{I_{\rm c} - I_\lambda}{I_{\rm c}}\right]_{\rm comp} * \frac{1}{\xi_{\rm macro}\sqrt{\pi}}\,\mathrm{e}^{-\xi^2/\xi_{\rm macro}^2}. \tag{3.81}$$

Inglis-Teller $N_{\rm e}$ estimate

$$\log N_{\rm e} = 23.2 - 7.5 \log n_{\rm max}^{\rm Balmer} \tag{3.82}$$

Two-Level Sequences



(a) thermal extinction = radiative excitation by a beam photon followed by collisional deexcitation,

(b) spontaneous scattering extinction = radiative excitation by a beam photon followed by spontaneous deexcitation,

(c) induced scattering extinction = radiative excitation by a beam photon followed by induced deexcitation,

(d) spontaneous thermal emission = collisional excitation followed by spontaneous emission of a photon into the beam,

(e) induced thermal emission = collisional excitation followed by induced emission of a photon into the beam,

(f) spontaneous scattering emission = radiative excitation followed by spontaneous emission of a photon into the beam,

(g) induced scattering emission = radiative excitation followed by induced emission of a photon into the beam.

Two-Level Sharp-Line Transport

Two-level transport equation

$$\begin{aligned}\frac{\mathrm{d}I_{\nu_0}}{\mathrm{d}s} \quad = \quad & - \underbrace{n_1\sigma^l_{\nu_0} I_{\nu_0}\frac{C_{21}}{P_{21}}}_{\text{(a)}} - \underbrace{n_1\sigma^l_{\nu_0} I_{\nu_0}\frac{A_{21}}{P_{21}}}_{\text{(b)}} - \underbrace{n_1\sigma^l_{\nu_0} I_{\nu_0}\frac{B_{21}J_{\nu_0}}{P_{21}}}_{\text{(c)}} \\ & + \underbrace{n_1 C_{12}\frac{(h\nu_0/4\pi)A_{21}}{P_{21}}}_{\text{(d)}} + \underbrace{n_1 C_{12}\frac{(h\nu_0/4\pi)B_{21}I_{\nu_0}}{P_{21}}}_{\text{(e)}} \\ & + \underbrace{n_1 B_{12}J_{\nu_0}\frac{(h\nu_0/4\pi)A_{21}}{P_{21}}}_{\text{(f)}} + \underbrace{n_1 B_{12}J_{\nu_0}\frac{(h\nu_0/4\pi)B_{21}I_{\nu_0}}{P_{21}}}_{\text{(g)}} \qquad (3.83)\end{aligned}$$

Total deexcitation probability

$$P_{21} \equiv A_{21} + B_{21}J_{\nu_0} + C_{21}. \qquad (3.84)$$

Two-level transport equation

$$\begin{aligned}\frac{\mathrm{d}I_{\nu_0}}{\mathrm{d}s} = \frac{h\nu_0}{4\pi}\, n_1 \Bigg[& - \underbrace{B_{12}I_{\nu_0}\frac{C_{21}}{P_{21}}}_{\text{(a)}} - \underbrace{B_{12}I_{\nu_0}\frac{A_{21}}{P_{21}}}_{\text{(b)}} - \underbrace{B_{12}I_{\nu_0}\frac{B_{21}J_{\nu_0}}{P_{21}}}_{\text{(c)}} \\ & + \underbrace{C_{12}\frac{A_{21}}{P_{21}}}_{\text{(d)}} + \underbrace{C_{12}\frac{B_{21}I_{\nu_0}}{P_{21}}}_{\text{(e)}} + \underbrace{B_{12}J_{\nu_0}\frac{A_{21}}{P_{21}}}_{\text{(f)}} + \underbrace{B_{12}J_{\nu_0}\frac{B_{21}I_{\nu_0}}{P_{21}}}_{\text{(g)}} \Bigg]. \qquad (3.85)\end{aligned}$$

Two-Level Sharp-Line Coefficients

Extinction coefficient

$$
\begin{aligned}
\alpha^l_{\nu_0} &= \frac{(\mathrm{a})-(\mathrm{e})+(\mathrm{b})+(\mathrm{c})-(\mathrm{g})}{I_{\nu_0}} \\
&= \frac{h\nu_0}{4\pi}\,n_1\,\Big[B_{12}\frac{C_{21}}{P_{21}}\,-\,C_{12}\frac{B_{21}}{P_{21}} \\
&\qquad +\,B_{12}\frac{A_{21}}{P_{21}}\,+\,B_{12}\frac{B_{21}J_{\nu_0}}{P_{21}}\,-\,B_{12}J_{\nu_0}\frac{B_{21}}{P_{21}}\Big] \\
&\equiv \alpha^{\mathrm{a}}_{\nu_0}\,+\,\alpha^{\mathrm{s}}_{\nu_0}.
\end{aligned}
\tag{3.86}
$$

Thermal part

$$
\begin{aligned}
\alpha^{\mathrm{a}}_{\nu_0} &= \frac{h\nu_0}{4\pi}\,n_1\,B_{12}\frac{C_{21}}{P_{21}}\left[1-\frac{C_{12}B_{21}}{C_{21}B_{12}}\right] \\
&= \frac{h\nu_0}{4\pi}\,n_1\,B_{12}\frac{C_{21}}{P_{21}}\left[1-\,\mathrm{e}^{-h\nu_0/kT}\right],
\end{aligned}
\tag{3.87}
$$

Scattering part

$$
\alpha^{s}_{\nu_0} = \frac{h\nu_0}{4\pi}\,n_1\,B_{12}\left[\frac{A_{21}}{P_{21}}+\frac{B_{21}J_{\nu_0}}{P_{21}}-J_{\nu_0}\frac{B_{21}}{P_{21}}\right] \tag{3.88}
$$

$$
= \frac{h\nu_0}{4\pi}\,n_1\,B_{12}\frac{A_{21}}{P_{21}}, \tag{3.89}
$$

Emission coefficient

$$
j^l_{\nu_0} = \frac{h\nu_0}{4\pi}\,n_1\,C_{12}\frac{A_{21}}{P_{21}}\,+\,\frac{h\nu_0}{4\pi}\,n_1\,B_{12}J_{\nu_0}\frac{A_{21}}{P_{21}} \tag{3.90}
$$

$$
\equiv j^{\mathrm{a}}_{\nu_0}\,+\,j^{\mathrm{s}}_{\nu_0}. \tag{3.91}
$$

Two-Level Sharp-Line Source Function

Thermal source function

$$\begin{aligned}
S^{\mathrm{a}}_{\nu_0} &\equiv j^{\mathrm{a}}_{\nu_0}/\alpha^{\mathrm{a}}_{\nu_0} \\
&= \frac{(h\nu_0/4\pi)\, n_1 C_{12} A_{21}/P_{21}}{(h\nu_0/4\pi)\, n_1\, B_{12} C_{21} \left[1 -\ \mathrm{e}^{-h\nu_0/kT}\right]/P_{21}} \\
&= \frac{A_{21}}{B_{12}} \frac{C_{12}}{C_{21}} \frac{\mathrm{e}^{h\nu_0/kT}}{\mathrm{e}^{h\nu_0/kT} - 1} \\
&= \frac{2h{\nu_0}^3}{c^2} \frac{1}{\mathrm{e}^{h\nu_0/kT} - 1} \\
&= B_{\nu_0}.
\end{aligned} \tag{3.92}$$

Scattering source function

$$\begin{aligned}
S^{\mathrm{s}}_{\nu_0} &\equiv j^{\mathrm{s}}_{\nu_0}/\alpha^{\mathrm{s}}_{\nu_0} \\
&= \frac{(h\nu_0/4\pi)\, n_1\, B_{12} J_{\nu_0} A_{21}/P_{21}}{(h\nu_0/4\pi)\, n_1\, B_{12} A_{21}/P_{21}} \\
&= J_{\nu_0}.
\end{aligned} \tag{3.93}$$

Collisional Destruction Probability

Combined line source function

$$S^l_{\nu_0} \equiv \frac{j^l_{\nu_0}}{\alpha^l_{\nu_0}} = (1 - \varepsilon_{\nu_0})\, J_{\nu_0} + \varepsilon_{\nu_0} B_{\nu_0}. \tag{3.94}$$

Destruction probability

$$\varepsilon_{\nu_0} \equiv \frac{\alpha^{\mathrm{a}}_{\nu_0}}{\alpha^{\mathrm{a}}_{\nu_0} + \alpha^{\mathrm{s}}_{\nu_0}} \tag{3.95}$$

$$= \frac{C_{21}\,[1 - \exp(-h\nu_0/kT)]}{C_{21}\,[1 - \exp(-h\nu_0/kT)] + A_{21}} \tag{3.96}$$

$$= \frac{C_{21}}{C_{21} + A_{21}/[1 - \exp(-h\nu_0/kT)]} \tag{3.97}$$

$$= \frac{C_{21}}{C_{21} + A_{21} + B_{21}B_{\nu_0}}. \tag{3.98}$$

Alternate form

$$\varepsilon'_{\nu_0} \equiv \alpha^{\mathrm{a}}_{\nu_0}/\alpha^{\mathrm{s}}_{\nu_0} \tag{3.99}$$

$$= \frac{\varepsilon_{\nu_0}}{1 - \varepsilon_{\nu_0}} \tag{3.100}$$

$$= \frac{C_{21}}{A_{21}}\,[1 - \mathrm{e}^{-h\nu_0/kT}], \tag{3.101}$$

Combined line source function

$$S^l_{\nu_0} = \frac{J_{\nu_0} + \varepsilon'_{\nu_0} B_{\nu_0}}{1 + \varepsilon'_{\nu_0}}. \tag{3.102}$$

Complete redistribution

Statistical equilibrium

$$\frac{\mathrm{d}n_2}{\mathrm{d}t} = n_1 P_{12} - n_2 P_{21} = 0. \tag{3.103}$$

Substitute

$$\begin{aligned} S^l_{\nu_0} &= \frac{2h\nu_0^3}{c^2} \frac{1}{\dfrac{g_2 n_1}{g_1 n_2} - 1} \\ &= \frac{(2h\nu_0^3/c^2)\left[(g_1/g_2)B_{12}\overline{J}^{\varphi}_{\nu_0} + (g_1/g_2)C_{12}\right]}{A_{21} + B_{21}\overline{J}^{\varphi}_{\nu_0} + C_{21} - (g_1/g_2)B_{12}\overline{J}^{\varphi}_{\nu_0} - (g_1/g_2)C_{12}}. \end{aligned}$$

Rework with Einstein relations

$$\begin{aligned} S^l_{\nu_0} &= \frac{(2h\nu_0^3/c^2)\left[(B_{21}/A_{21})\overline{J}^{\varphi}_{\nu_0} + (C_{21}/A_{21})\exp(-h\nu_0/kT)\right]}{1 + C_{21}/A_{21} - (C_{21}/A_{21})\exp(-h\nu_0/kT)} \\ &= \frac{\overline{J}^{\varphi}_{\nu_0} + (C_{21}/A_{21})(2h\nu_0^3/c^2)\exp(-h\nu_0/kT)}{1 + C_{21}/A_{21} - (C_{21}/A_{21})\exp(-h\nu_0/kT)}. \end{aligned}$$

Rework with Einstein relations

$$S^l_{\nu_0} = \frac{\overline{J}^{\varphi}_{\nu_0} + \varepsilon'_{\nu_0} B_{\nu_0}}{1 + \varepsilon'_{\nu_0}} \tag{3.104}$$

$$= (1 - \varepsilon_{\nu_0})\,\overline{J}^{\varphi}_{\nu_0} + \varepsilon_{\nu_0} B_{\nu_0}. \tag{3.105}$$

Frequency dependence

$$S^{\mathrm{tot}}_{\nu} = \frac{\alpha^l_{\nu} S^l_{\nu_0} + \alpha^c_{\nu} S^c_{\nu}}{\alpha^l_{\nu} + \alpha^c_{\nu}} \tag{3.106}$$

Bound-free destruction

$$\varepsilon^{\mathrm{bf}}_{\nu_0} = \frac{C_{ci}}{C_{ci} + R^{\mathrm{spon}}_{ci} + \left[R^{\mathrm{ind}}_{ci}\right]_{\mathrm{LTE}}}, \tag{3.107}$$

Edge averages

$$\overline{S}^{\sigma}_{\nu_0} \equiv \frac{\int_{\nu_0}^{\infty} S_{\nu}\,\sigma_{ic}(\nu)\,\mathrm{d}\nu}{\int_{\nu_0}^{\infty} \sigma_{ic}(\nu)\,\mathrm{d}\nu} \qquad \overline{J}^{\sigma}_{\nu_0} \equiv \frac{\int_{\nu_0}^{\infty} J_{\nu}\,\sigma_{ic}(\nu)\,\mathrm{d}\nu}{\int_{\nu_0}^{\infty} \sigma_{ic}(\nu)\,\mathrm{d}\nu} \qquad \overline{B}^{\sigma}_{\nu_0} \equiv \frac{\int_{\nu_0}^{\infty} B_{\nu}\,\sigma_{ic}(\nu)\,\mathrm{d}\nu}{\int_{\nu_0}^{\infty} \sigma_{ic}(\nu)\,\mathrm{d}\nu} \tag{3.108}$$

One-level-plus-continuum atoms

$$\overline{S}^{\sigma}_{\nu_0} = (1 - \varepsilon^{\mathrm{bf}}_{\nu_0})\,\overline{J}^{\sigma}_{\nu_0} + \varepsilon^{\mathrm{bf}}_{\nu_0}\overline{B}^{\sigma}_{\nu_0}. \tag{3.109}$$

Chapter 4

Transport Equation

General

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \frac{\partial I_\nu}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial s} = \frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = j_\nu - \alpha_\nu I_\nu \tag{4.1}$$

$$\frac{\partial I_\nu}{\partial s} = \frac{\partial I_\nu}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}s}. \tag{4.2}$$

Spherical geometry

$$\frac{\partial I_\nu}{\partial s} = \cos\theta\,\frac{\partial I_\nu}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial I_\nu}{\partial \theta} = \mu\,\frac{\partial I_\nu}{\partial r} + \frac{1-\mu^2}{r}\,\frac{\partial I_\nu}{\partial \mu} = j_\nu - \alpha_\nu I_\nu \tag{4.3}$$

$$\frac{\mu}{\kappa_\nu\rho}\,\frac{\partial I_\nu}{\partial r} + \frac{1-\mu^2}{\kappa_\nu\rho r}\frac{\partial I_\nu}{\partial \mu} = S_\nu - I_\nu. \tag{4.4}$$

Plane parallel layers

$$\mu\,\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu. \tag{4.5}$$

Moment versions

$$\begin{aligned} \frac{1}{2}\int_{-1}^{+1}\mu\,\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu}\mathrm{d}\mu &= \frac{1}{2}\int_{-1}^{+1} I_\nu\,\mathrm{d}\mu - \frac{1}{2}\int_{-1}^{+1} S_\nu\,\mathrm{d}\mu \\ \frac{\mathrm{d}H_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} &= J_\nu(\tau_\nu) - S_\nu(\tau_\nu) \end{aligned} \tag{4.6}$$

$$-\frac{\mathrm{d}H_\nu(z)}{\mathrm{d}z} = \kappa_\nu\rho\,J_\nu(z) - \kappa_\nu\rho S_\nu(z) \tag{4.7}$$

$$\begin{aligned} \frac{1}{2}\int_{-1}^{+1}\mu^2\,\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu}\mathrm{d}\mu &= \frac{1}{2}\int_{-1}^{+1}\mu\,I_\nu\,\mathrm{d}\mu - \frac{1}{2}\int_{-1}^{+1}\mu\,S_\nu\,\mathrm{d}\mu \\ \frac{\mathrm{d}K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} &= H_\nu(\tau_\nu) \end{aligned} \tag{4.8}$$

$$\frac{\mathrm{d}^2K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu). \tag{4.9}$$

Moments with $E_n(x)$

Formal solution

$$I_\nu^+(\tau_\nu,\mu) \;=\; +\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/\mu \tag{4.10}$$

$$I_\nu^-(\tau_\nu,\mu) \;=\; +\int_{0}^{\tau_\nu} S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/|\mu|, \tag{4.11}$$

Exponential integrals

$$E_n(x) \equiv \int_1^\infty \frac{\mathrm{e}^{-xw}}{w^n}\,\mathrm{d}w = \int_0^1 \,\mathrm{e}^{-x/\mu}\,\mu^{n-1}\,\frac{\mathrm{d}\mu}{\mu} \tag{4.12}$$

Asymptotic value

$$E_n(x) = \frac{\mathrm{e}^{-x}}{x}\left[1 - \frac{n}{x} + \frac{n(n+1)}{x^2} + \cdots\right] \tag{4.13}$$

Schwarzschild equation

$$\begin{aligned}
J_\nu(\tau_\nu) &\equiv \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu,\mu)\,\mathrm{d}\mu \\
&= \frac{1}{2}\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\,E_1(t_\nu - \tau_\nu)\,\mathrm{d}t_\nu + \frac{1}{2}\int_0^{\tau_\nu} S_\nu(t_\nu)\,E_1(\tau_\nu - t_\nu)\,\mathrm{d}t_\nu \\
&= \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\,E_1(|t_\nu - \tau_\nu|)\,\mathrm{d}t_\nu,
\end{aligned} \tag{4.14}$$

Milne equation

$$\begin{aligned}
\mathcal{F}_\nu(\tau_\nu) &= \mathcal{F}_\nu^+(\tau_\nu) - \mathcal{F}_\nu^-(\tau_\nu) \\
&= 2\pi\int_0^1 \mu I_\nu(\tau_\nu)\,\mathrm{d}\mu \;-\; 2\pi\int_0^{-1} \mu I_\nu(\tau_\nu)\,\mathrm{d}\mu \\
&= 2\pi\int_{\tau_\nu}^\infty S_\nu(t_\nu)\,E_2(t_\nu - \tau_\nu)\,\mathrm{d}t_\nu - 2\pi\int_0^{\tau_\nu} S_\nu(t_\nu)\,E_2(\tau_\nu - t_\nu)\,\mathrm{d}t_\nu,
\end{aligned} \tag{4.15}$$

K integral

$$K_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\,E_3(|t_\nu - \tau_\nu|)\,\mathrm{d}t_\nu. \tag{4.16}$$

Emergent intensity and flux

$$I_\nu^+(0,\mu) \;=\; \int_0^\infty S_\nu(\tau_\nu)\,\mathrm{e}^{-\tau_\nu/\mu}\,\mathrm{d}\tau_\nu/\mu \tag{4.17}$$

$$\mathcal{F}_\nu^+(0) \;=\; 2\pi\int_0^\infty S_\nu(\tau_\nu)\,E_2(\tau_\nu)\,\mathrm{d}\tau_\nu. \tag{4.18}$$

Operators

Laplace operator

$$\mathcal{L}_{1/\mu}\{S_\nu(\tau_\nu)\} \equiv \int_0^\infty S_\nu(\tau_\nu)\,\mathrm{e}^{-\tau_\nu/\mu}\,\mathrm{d}\tau_\nu/\mu = I_\nu^+(0,\mu). \tag{4.19}$$

Lambda operator

$$\boldsymbol{\Lambda}_\tau[f(t)] \equiv \frac{1}{2}\int_0^\infty f(t)\,E_1(|t-\tau|)\,\mathrm{d}t. \tag{4.20}$$

Properties

$$\begin{aligned}
\boldsymbol{\Lambda}_\tau[1] &= 1-\frac{1}{2}E_2(\tau)\\
\boldsymbol{\Lambda}_\tau[t] &= \tau+\frac{1}{2}E_3(\tau)\\
\boldsymbol{\Lambda}_\tau[t^2] &= \frac{2}{3}+\tau^2-E_4(\tau)\\
\boldsymbol{\Lambda}_\tau[t^p] &= \frac{1}{2}p!\left[\sum_{k=0}^{p}\frac{\tau^k}{k!}\delta_\alpha+(-1)^{p+1}E_{p+2}(\tau)\right]
\end{aligned}$$

Schwarzschild equation

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\,E_1(|t_\nu-\tau_\nu|)\,\mathrm{d}t_\nu = \boldsymbol{\Lambda}_{\tau_\nu}[S_\nu(t_\nu)]. \tag{4.21}$$

Phi operator

$$\begin{aligned}
\boldsymbol{\Phi}_{\tau_\nu}[S_\nu(t_\nu)] &\equiv 2\int_{\tau_\nu}^\infty S_\nu(t_\nu)\,E_2(t_\nu-\tau_\nu)\,\mathrm{d}t_\nu - 2\int_0^{\tau_\nu} S_\nu(t_\nu)\,E_2(\tau_\nu-t_\nu)\,\mathrm{d}t_\nu \qquad (4.22)\\
&= F_\nu(\tau_\nu)
\end{aligned}$$

Chi operator

$$\begin{aligned}
\chi_{\tau_\nu}[S_\nu(t_\nu)] &\equiv 2\int_0^\infty S_\nu(t_\nu)\,E_3(|t_\nu-\tau_\nu|)\,\mathrm{d}t_\nu \qquad (4.23)\\
&= 4\,K_\nu(\tau_\nu),
\end{aligned}$$

$$\boldsymbol{\Phi}_\tau[f(t)] = \frac{\mathrm{d}}{\mathrm{d}\tau}\chi_\tau[f(t)]. \tag{4.24}$$

Generalized Lambda Operators

Hubený notation

$$
\begin{aligned}
I_\nu(\tau_\nu, \mu) &= \boldsymbol{\Lambda}_{\mu\nu}[S_\nu(t_\nu)] && (4.25) \\
J_\nu(\tau_\nu) &= \boldsymbol{\Lambda}_{\nu}[S_\nu(t_\nu)], && (4.26)
\end{aligned}
$$

Angle-averaged $\boldsymbol{\Lambda}_\nu$

$$
\boldsymbol{\Lambda}_\nu = \frac{1}{2}\int_{-1}^{+1} \boldsymbol{\Lambda}_{\mu\nu}\,\mathrm{d}\mu. \tag{4.27}
$$

Slow $\boldsymbol{\Lambda}_{\mu\nu}$ implementation

$$
\begin{aligned}
\boldsymbol{\Lambda}_{+\mu\nu}[S_\nu] = I^+_\nu(\tau_\nu, \mu) &= \int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/\mu && (4.28) \\
&= \mathrm{e}^{\tau_{\nu\mu}} \int_{\tau_{\nu\mu}}^{\infty} S_\nu\,\mathrm{e}^{-t_{\nu\mu}}\,\mathrm{d}t_{\nu\mu} && (4.29)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Lambda}_{-\mu\nu}[S_\nu] = I^-_\nu(\tau_\nu, -|\mu|) &= \int_{0}^{\tau_\nu} S_\nu(t_\nu)\,\mathrm{e}^{-(\tau_\nu-t_\nu)/|\mu|}\,\mathrm{d}t_\nu/|\mu| && (4.30) \\
&= \mathrm{e}^{-\tau_{\nu\mu}} \int_{0}^{\tau_{\nu\mu}} S_\nu\,\mathrm{e}^{t_{\nu\mu}}\,\mathrm{d}t_{\nu\mu}, && (4.31)
\end{aligned}
$$

Surface Approximations

Polynomial expansion

$$S_\nu(\tau_\nu) = \sum_{n=0}^{\infty} a_n {\tau_\nu}^n$$

Eddington-Barbier approximations

$$\begin{aligned}
I_\nu^+(0,\mu) &\approx a_0 + a_1\mu \\
&\approx S_\nu(\tau_\nu = \mu), \qquad (4.32) \\
J_\nu(0) &\approx a_0 + \frac{2a_2}{3} - \frac{a_0}{2} + \frac{a_1}{4} - \frac{a_2}{3} \\
&\approx \frac{a_0}{2} + \frac{a_1}{4} + \frac{a_2}{3} \\
&\approx \frac{1}{2} S_\nu(\tau_\nu = 1/2), \qquad (4.33) \\
F_\nu(0) &= a_0 + \frac{2}{3}a_1 + a_2 + \cdots \\
&\approx S_\nu(\tau_\nu = \frac{2}{3}), \qquad (4.34)
\end{aligned}$$

Second Eddington approximation

$$\begin{aligned}
F_\nu(0) &\equiv 2\int_{-1}^{+1} I_\nu(0,\mu)\mu \,\mathrm{d}\mu \\
&= 2\int_0^1 I_\nu(0,\mu)\mu \,\mathrm{d}\mu \\
&\approx 2 < I_\nu^+(0,\mu) > \int_0^1 \mu \,\mathrm{d}\mu \\
&\approx 2\, J_\nu(0), \qquad (4.35)
\end{aligned}$$

Error for linear S

$$\frac{F_\nu(0)}{2\,J_\nu(0)} = \frac{a_0/2 + a_1/3}{a_0/2 + a_1/4} \neq 1. \qquad (4.36)$$

Taylor Expansions

$$S_\nu(\tau_\nu) = \sum_{n=0}^{\infty} \frac{(t_\nu - \tau_\nu)^n}{n!} \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \tag{4.37}$$

$\tau_\nu > 1$

$$I_\nu(\tau_\nu, \mu) = S_\nu(\tau_\nu) + \mu \left[\frac{\mathrm{d}S_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} + \mu^2 \left[\frac{\mathrm{d}^2 S_\nu(t_\nu)}{\mathrm{d}t_\nu^2}\right]_{\tau_\nu} + \cdots \tag{4.38}$$

$$J_\nu(\tau_\nu) = \frac{1}{2}\sum_{n=0}^{\infty} \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \int_{-1}^{+1} \mu^n \,\mathrm{d}\mu = \sum_{k=0}^{\infty} \frac{1}{2k+1} \left[\frac{\mathrm{d}^{(2k)} S_\nu(t_\nu)}{\mathrm{d}t_\nu^{(2k)}}\right]_{\tau_\nu}$$

$$J_\nu(\tau_\nu) = S_\nu(\tau_\nu) + \frac{1}{3}\left[\frac{\mathrm{d}^2 S_\nu(t_\nu)}{\mathrm{d}t_\nu^2}\right]_{\tau_\nu} + \cdots \tag{4.39}$$

$$F_\nu(\tau_\nu) = \frac{4}{3}\left[\frac{\mathrm{d}S_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} + \frac{4}{5}\left[\frac{\mathrm{d}^3 S_\nu(t_\nu)}{\mathrm{d}t_\nu^3}\right]_{\tau_\nu} + \cdots \tag{4.40}$$

$$K_\nu(\tau_\nu) = \frac{1}{3}S_\nu(\tau_\nu) + \frac{1}{5}\left[\frac{\mathrm{d}^2 S_\nu(t_\nu)}{\mathrm{d}t_\nu^2}\right]_{\tau_\nu} + \cdots \tag{4.41}$$

Convergence

$$\left|\frac{\mathrm{d}^n S_\nu}{\mathrm{d}t_\nu^n}\right| \sim \frac{S_\nu}{t_\nu^n}$$

$$\frac{\left|\mathrm{d}^{n+2} S_\nu/\mathrm{d}t_\nu^{n+2}\right|}{\left|\mathrm{d}^n S_\nu/\mathrm{d}t_\nu^n\right|} \sim \frac{S_\nu/t_\nu^{n+2}}{S_\nu/t_\nu^n} \sim \frac{1}{t_\nu^2},$$

$\tau_\nu \gg 1$

$$I_\nu(\tau_\nu, \mu) \approx S_\nu(\tau_\nu) + \mu \left[\frac{\mathrm{d}S_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} \tag{4.42}$$

$$J_\nu(\tau_\nu) \approx S_\nu(\tau_\nu) \tag{4.43}$$

$$F_\nu(\tau_\nu) \approx \frac{4}{3}\left[\frac{\mathrm{d}S_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} \tag{4.44}$$

$$K_\nu(\tau_\nu) \approx \frac{1}{3}S_\nu(\tau_\nu). \tag{4.45}$$

Relative anisotropy

$$\frac{\mathrm{d}S_\nu/\mathrm{d}\tau_\nu}{S_\nu} \sim \frac{S_\nu/\tau_\nu}{S_\nu} \sim \frac{1}{\tau_\nu}.$$

Diffusion Approximation

Linear anisotropy and LTE

$$I_\nu(\tau_\nu, \mu) \approx B_\nu(\tau_\nu) + \mu \left[\frac{\mathrm{d}B_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} \tag{4.46}$$

$$J_\nu(z) \approx B_\nu(z) \tag{4.47}$$

$$\mathcal{F}_\nu(z) \approx 2\pi \int_{-1}^{+1} \mu \, I_\nu \, \mathrm{d}\mu \approx \frac{4\pi}{3} \frac{\mathrm{d}B_\nu(z)}{\mathrm{d}\tau_\nu}. \tag{4.48}$$

Rosseland mean opacity

$$\frac{1}{\alpha_\mathrm{R}} \equiv \frac{\int_0^\infty (1/\alpha_\nu)(\mathrm{d}B_\nu/\mathrm{d}T)\, \mathrm{d}\nu}{\int_0^\infty (\mathrm{d}B_\nu/\mathrm{d}T)\, \mathrm{d}\nu} \tag{4.49}$$

$$\frac{1}{\kappa_\mathrm{R}} \equiv \frac{\int_0^\infty (1/\kappa_\nu)(\mathrm{d}B_\nu/\mathrm{d}T)\, \mathrm{d}\nu}{\int_0^\infty (\mathrm{d}B_\nu/\mathrm{d}T)\, \mathrm{d}\nu} \tag{4.50}$$

Diffusion/conduction/Rosseland approximation

$$\begin{aligned}
\mathcal{F}(z) &\equiv \int_0^\infty \mathcal{F}_\nu(z)\, \mathrm{d}\nu \\
&\approx -\frac{4\pi}{3} \int_0^\infty \frac{1}{\alpha_\nu} \frac{\mathrm{d}B_\nu}{\mathrm{d}z}\, \mathrm{d}\nu \\
&\approx -\frac{4\pi}{3} \int_0^\infty \frac{1}{\alpha_\nu} \frac{\mathrm{d}B_\nu}{\mathrm{d}T} \frac{\mathrm{d}T}{\mathrm{d}z}\, \mathrm{d}\nu && (4.51) \\
&\approx -\frac{16}{3} \frac{\sigma T^3}{\alpha_\mathrm{R}} \frac{\mathrm{d}T}{\mathrm{d}z} && (4.52) \\
&\approx -\frac{1}{3} \frac{c}{\kappa_\mathrm{R}\rho} \frac{\mathrm{d}u}{\mathrm{d}z} && (4.53)
\end{aligned}$$

Shu (1991):

> A “random walk” slows down the free-flight speed c by a typical factor of $l/R_\odot$, so that the time $R_\odot^2/\mathcal{D}$ (with the radiative diffusivity $\mathcal{D} = c/3\rho\kappa_\mathrm{R}$) is roughly $3R_\odot/l$ times longer than the free-flight time $R_\odot/c$ of about 2 s. This process prevents the Sun from releasing its considerable internal reservoir of photons in one powerful blast, but instead regulates it to the stately observed luminosity of $L_\odot = 3.86 \times 10^{33}\,\mathrm{erg\,s^{-1}}$. In any case, apart from being useful for rough order-of-magnitude arguments, this accurate equation constitutes one of the fundamental equations underlying the whole theory of stellar structure and evolution.

(First) Eddington Approximation

Flux and mean intensity

$$K_\nu(\tau_\nu) \approx \frac{1}{3} J_\nu(\tau_\nu). \tag{4.54}$$

Derivation

$$\begin{aligned}
K_\nu(\tau_\nu) &\equiv \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu,\mu)\mu^2\,\mathrm{d}\mu \\
&\approx \frac{1}{2} < I_\nu(\tau_\nu,\mu) > \int_{-1}^{+1} \mu^2\,\mathrm{d}\mu \\
&\approx \frac{1}{3}\,J_\nu(\tau_\nu)
\end{aligned}$$

Linear anisotropy

$$I_\nu(\tau_\nu,\mu) \equiv a_0(\tau_\nu) + a_1(\tau_\nu)\,\mu \tag{4.55}$$

$$J_\nu(\tau_\nu) \equiv \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu,\mu)\,\mathrm{d}\mu = a_0(\tau_\nu), \tag{4.56}$$

$$H_\nu(\tau_\nu) \equiv \frac{1}{2}\int_{-1}^{+1} \mu I_\nu(\tau_\nu,\mu)\,\mathrm{d}\mu = a_1(\tau_\nu)/3, \tag{4.57}$$

$$K_\nu(\tau_\nu) \equiv \frac{1}{2}\int_{-1}^{+1} \mu^2 I_\nu(\tau_\nu,\mu)\,\mathrm{d}\mu = a_0(\tau_\nu)/3. \tag{4.58}$$

Second-order transport equation

$$\frac{1}{3}\frac{\mathrm{d}^2 J_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu). \tag{4.59}$$

Coherent scattering

$$S_\nu = (1-\varepsilon_\nu)J_\nu + \varepsilon_\nu B_\nu, \tag{4.60}$$

$$\frac{1}{3}\frac{\mathrm{d}^2 J_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = \varepsilon_\nu\left[J_\nu(\tau_\nu) - B_\nu(\tau_\nu)\right]. \tag{4.61}$$

Illustration: Scattering Atmosphere

Linear B_ν

$$B_\nu(\tau_\nu) \equiv B_{\nu,0} + b_\nu\, \tau_\nu \tag{4.62}$$

Eddington approximation

$$\frac{1}{3}\frac{\mathrm{d}^2}{\mathrm{d}\tau_\nu^2}(J_\nu - B_\nu) = \varepsilon_\nu (J_\nu - B_\nu). \tag{4.63}$$

Solution for constant ε_ν

$$J_\nu - B_\nu = C_1\, \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\,\tau_\nu} + C_2\, \mathrm{e}^{+\sqrt{3\varepsilon_\nu}\,\tau_\nu}. \tag{4.64}$$

Boundary conditions

$$\begin{aligned} J_\nu(0) &= B_{\nu,0} + C_1 \\ &= a_\nu H_\nu(0) = a_\nu \left[\frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu}\right]_{\tau_\nu=0} = (a_\nu/3) \left[\frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}\right]_{\tau_\nu=0} = -(a_\nu/3)\sqrt{3\varepsilon_\nu}\, C_1 + a_\nu b_\nu/3 \\ C_1 &= -\frac{B_{\nu,0} - a_\nu b_\nu/3}{1 + (a_\nu/3)\sqrt{3\varepsilon_\nu}}. \end{aligned}$$

Results

$$\begin{aligned} J_\nu(\tau_\nu) &= B_\nu(\tau_\nu) + C_1\, \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\,\tau_\nu} \\ &= B_{\nu,0} + b_\nu\tau_\nu - \frac{B_{\nu,0} - a_\nu b_\nu/3}{1 + (a_\nu/3)\sqrt{3\varepsilon_\nu}}\, \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\,\tau_\nu} \end{aligned} \tag{4.65}$$

$$\begin{aligned} S_\nu(\tau_\nu) &= B_\nu(\tau_\nu) + (1-\varepsilon_\nu) C_1\, \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\,\tau_\nu} \\ &= B_{\nu,0} + b_\nu\tau_\nu - (1-\varepsilon_\nu)\frac{B_{\nu,0} - a_\nu b_\nu/3}{1 + (a_\nu/3)\sqrt{3\varepsilon_\nu}}\, \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\,\tau_\nu} \end{aligned} \tag{4.66}$$

$$H_\nu(\tau_\nu) = b_\nu/3 + \sqrt{\varepsilon_\nu}\, \frac{B_{\nu,0} - a_\nu b_\nu/3}{\sqrt{3} + a_\nu\sqrt{\varepsilon_\nu}}\, \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\,\tau_\nu} \tag{4.67}$$

$$I_\nu^+(0,\mu) = B_{\nu,0} + b_\nu\mu - \frac{(1-\varepsilon_\nu)(B_{\nu,0} - a_\nu b_\nu/3)}{(1 + (a_\nu/3)\sqrt{3\varepsilon_\nu})(1 + \mu\,\sqrt{3\varepsilon_\nu})}, \tag{4.68}$$

Isothermal Atmosphere

$$\begin{aligned}
J_\nu(\tau_\nu) &= B_{\nu,0}\left[1 - \frac{1}{1+(a_\nu/3)\sqrt{3\varepsilon_\nu}}\,\mathrm{e}^{-\sqrt{3\varepsilon_\nu}\,\tau_\nu}\right] && (4.69)\\
S_\nu(\tau_\nu) &= B_{\nu,0}\left[1 - \frac{1-\varepsilon_\nu}{1+(a_\nu/3)\sqrt{3\varepsilon_\nu}}\,\mathrm{e}^{-\sqrt{3\varepsilon_\nu}\,\tau_\nu}\right] && (4.70)\\
I_\nu^+(0,\mu) &= B_{\nu,0}\left[1 - \frac{1-\varepsilon_\nu}{(1+(a_\nu/3)\sqrt{3\varepsilon_\nu})(1+\mu\,\sqrt{3\varepsilon_\nu})}\right] && (4.71)
\end{aligned}$$

$\varepsilon = 1$

$$\begin{aligned}
J_\nu(\tau_\nu) &= \left[1 - \frac{\mathrm{e}^{-\sqrt{3}\,\tau_\nu}}{1+(a_\nu/3)\sqrt{3}}\right] B_{\nu,0} = \left[1-(1/2)\,\mathrm{e}^{-\sqrt{3}\,\tau_\nu}\right] B_{\nu,0} && (4.72)\\
S_\nu(\tau_\nu) &= B_{\nu,0} && (4.73)\\
H_\nu(\tau_\nu) &= \frac{\mathrm{e}^{-\sqrt{3}\,\tau_\nu}}{\sqrt{3}+a_\nu}\,B_{\nu,0} = \frac{\mathrm{e}^{-\sqrt{3}\,\tau_\nu}}{2\sqrt{3}}\,B_{\nu,0} && (4.74)\\
I_\nu^+(0,\mu) &= B_{\nu,0} && (4.75)\\
\mathcal{F}_\nu(0) &= \frac{2\pi}{\sqrt{3}}\,B_{\nu,0}. && (4.76)
\end{aligned}$$

$a_\nu = \sqrt{3}$

$$\begin{aligned}
J_\nu(\tau_\nu) &= \left[1 - \frac{1}{1+\sqrt{\varepsilon_\nu}}\,\mathrm{e}^{-\tau_\nu^*}\right] B_{\nu,0} && (4.77)\\
S_\nu(\tau_\nu) &= \left[1 - (1-\sqrt{\varepsilon_\nu})\,\mathrm{e}^{-\tau_\nu^*}\right] B_{\nu,0} && (4.78)\\
H_\nu(\tau_\nu) &= \frac{1}{\sqrt{3}}\,\frac{\sqrt{\varepsilon_\nu}}{1+\sqrt{\varepsilon_\nu}}\;\mathrm{e}^{-\tau_\nu^*}\,B_{\nu,0}. && (4.79)
\end{aligned}$$

Surface values

$$\begin{aligned}
J_\nu(0) &= \frac{\sqrt{\varepsilon_\nu}}{1+\sqrt{\varepsilon_\nu}}\,B_{\nu,0} && (4.80)\\
S_\nu(0) &= \sqrt{\varepsilon_\nu}\,B_{\nu,0} && (4.81)\\
I_\nu^+(0,\mu) &= \frac{1+\mu\sqrt{3}}{1+\mu\sqrt{3\varepsilon_\nu}}\,\sqrt{\varepsilon_\nu}\,B_{\nu,0} && (4.82)\\
\mathcal{F}_\nu^+(0) &= \frac{4\pi}{\sqrt{3}}\,\frac{\sqrt{\varepsilon_\nu}}{1+\sqrt{\varepsilon_\nu}}\,B_{\nu,0}. && (4.83)
\end{aligned}$$

$\varepsilon_\nu \ll 1$

$$I_\nu^+(0,1) \approx (1+\sqrt{3})\sqrt{\varepsilon_\nu}\, B_{\nu,0} = 2.7\sqrt{\varepsilon_\nu}\, B_{\nu,0} \ll B_{\nu,0} \tag{4.84}$$

$$\mathcal{F}_\nu^+(0) \approx \frac{4\pi}{\sqrt{3}}\sqrt{\varepsilon_\nu}\, B_{\nu,0} = 7.3\sqrt{\varepsilon_\nu}\, B_{\nu,0} \ll B_{\nu,0}. \tag{4.85}$$

Thermalization depth

$$\Lambda_\nu = 1/\sqrt{\varepsilon_\nu} \tag{4.86}$$

$J_\nu - B_\nu$ split

$$\frac{J_\nu(\tau_\nu) - B_\nu(\tau_\nu)}{B_{0,\nu}} = \frac{a_\nu b_\nu/3B_{0,\nu} - 1}{1+(a_\nu/3)\sqrt{3\varepsilon_\nu}}\ \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\,\tau_\nu}, \tag{4.87}$$

$\varepsilon_\nu \ll 1$

$$J_\nu(\tau_\nu) - B_\nu(\tau_\nu) \approx [a_\nu b_\nu/3 - B_{\nu,0}]\ \mathrm{e}^{-\tau_\nu^*} \tag{4.88}$$

RE gradient

$$B_\nu(\tau_\nu^c) \approx B_{\nu,0}\,(1 + 1.5\,\tau_\nu^c) \tag{4.89}$$

RE gradient as seen by line

$$B_\nu(\tau_\nu^{\mathrm{tot}}) \approx B_{\nu,0}\left(1 + \frac{1.5}{1+\eta_\nu}\,\tau_\nu^{\mathrm{tot}}\right). \tag{4.90}$$

Coherent Scattering with Thermal Background

Continuum destruction probability

$$\delta_\nu \equiv \frac{\alpha_\nu^c}{\alpha_\nu^l + \alpha_\nu^c}. \tag{4.91}$$

Extinction ratio

$$\eta_\nu \equiv \frac{\alpha_\nu^l}{\alpha_\nu^c} \tag{4.92}$$

Combined destruction probability

$$\lambda_\nu = \delta_\nu + (1-\delta_\nu)\,\varepsilon_\nu = \varepsilon_\nu + (1-\varepsilon_\nu)\,\delta_\nu = \frac{\alpha_\nu^c + \varepsilon_\nu \alpha_\nu^l}{\alpha_\nu^c + \alpha_\nu^l} = \frac{1+\varepsilon_\nu \eta_\nu}{1+\eta_\nu}, \tag{4.93}$$

Total source function

$$\begin{aligned} S_\nu^{\rm tot} &= \frac{S_\nu^c + \eta_\nu S_\nu^l}{1+\eta_\nu} = \frac{B_\nu + \eta_\nu\left[(1-\varepsilon_\nu)J_\nu + \varepsilon_\nu B_\nu\right]}{1+\eta_\nu} \\ &= (1-\lambda_\nu)J_\nu + \lambda_\nu B_\nu \qquad (4.94) \\ &= (1-\lambda_\nu)J_\nu + (\lambda_\nu - \varepsilon_\nu)B_\nu + \varepsilon_\nu B_\nu. \qquad (4.95) \end{aligned}$$

Thermalization depth for coherent scattering

$$\Lambda_\nu = \frac{1}{\sqrt{\lambda_\nu}} = \sqrt{\frac{1+\eta_\nu}{1+\varepsilon_\nu \eta_\nu}}, \tag{4.96}$$

Corresponding continuum optical depth

$$\tau_\nu^c(\Lambda_\nu) = \frac{1}{1+\eta_\nu}\Lambda_\nu = \frac{1}{\left[(1+\eta_\nu)(1+\varepsilon_\nu\eta_\nu)\right]^{1/2}}, \tag{4.97}$$

$\varepsilon \approx 1$

$$\tau_\nu^c(\Lambda_\nu) \approx 1/\eta_\nu \approx \tau_\nu^c[\tau_\nu^{\rm tot}\!=\!1] \tag{4.98}$$

$\varepsilon_\nu \ll 1/\eta_\nu$

$$\tau_\nu^c(\Lambda_\nu) \approx 1/\sqrt{\eta_\nu} \approx \tau_\nu^c[\tau_\nu^{\rm tot}\!=\!\sqrt{\eta_\nu}]. \tag{4.99}$$

Thomson and Rayleigh scattering

$$\varepsilon_\nu \equiv \frac{\sum \alpha_\nu^{\rm bf} + \sum \alpha_\nu^{\rm ff}}{\alpha_\nu^{\rm T} + \alpha_\nu^{\rm R} + \sum \alpha_\nu^{\rm bf} + \sum \alpha_\nu^{\rm ff}}, \tag{4.100}$$

Complete Redistribution

Continuum destruction probability

$$\delta_{\nu_0} \equiv \int_0^\infty \delta_\nu \, \varphi(\nu - \nu_0) \, \mathrm{d}\nu = \int_0^\infty \frac{\varphi(\nu - \nu_0)}{1 + \eta_{\nu_0}\varphi(\nu - \nu_0)} \, \mathrm{d}\nu = \frac{1}{\eta_{\nu_0}} \int_0^\infty \frac{\eta_{\nu_0}\varphi(\nu - \nu_0)}{1 + \eta_{\nu_0}\varphi(\nu - \nu_0)} \, \mathrm{d}\nu, \tag{4.101}$$

Total destruction probability

$$\lambda_{\nu_0} \equiv \int_0^\infty \lambda_\nu \varphi(\nu - \nu_0) \, \mathrm{d}\nu = \varepsilon_{\nu_0} + (1 - \varepsilon_{\nu_0}) \, \delta_{\nu_0} \tag{4.102}$$

Total source function

$$S_{\nu_0}^{\mathrm{tot}} = (1 - \lambda_{\nu_0})\overline{J}_{\nu_0} + \lambda_{\nu_0} B_{\nu_0}. \tag{4.103}$$

Second-order transport equation

$$\frac{1}{3} \frac{\mathrm{d}^2 J_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = J_\nu(\tau_\nu) - (1 - \lambda_{\nu_0})\overline{J}_{\nu_0} - \lambda_{\nu_0} B_{\nu_0}, \tag{4.104}$$

Thermalization depths

$$\Lambda_{\nu_0} = 1/\varepsilon_{\nu_0}^{1/2} \quad \text{delta function} \tag{4.105}$$
$$\Lambda_{\nu_0} \approx 1/\varepsilon_{\nu_0} \quad \text{Gauss profile} \tag{4.106}$$
$$\Lambda_{\nu_0} \approx 1/\varepsilon_{\nu_0}^2 \quad \text{Lorentz profile.} \tag{4.107}$$

With background continuum

$$\Lambda_{\nu_0} = 1/\lambda_{\nu_0}^{1/2} \quad \text{delta function} \tag{4.108}$$
$$\Lambda_{\nu_0} \approx 1/\lambda_{\nu_0} \quad \text{Gauss profile} \tag{4.109}$$
$$\Lambda_{\nu_0} \approx 1/\lambda_{\nu_0}^2 \quad \text{Lorentz profile,} \tag{4.110}$$

Chapter 5

Numerical Solutions

Angle discretization

$$\begin{aligned} J &\equiv \frac{1}{2}\int_{-1}^{+1} I\,\mathrm{d}\mu \\ &= \frac{1}{2}\int_{0}^{+1} I^{+}\,\mathrm{d}\mu + \frac{1}{2}\int_{-1}^{0} I^{-}\,\mathrm{d}\mu \\ &= \frac{1}{2}\int_{0}^{+1} I^{+}\,\mathrm{d}\mu + \frac{1}{2}\int_{0}^{+1} I^{-}\,\mathrm{d}(-\mu) \qquad (5.1) \\ &\approx \frac{1}{2}\sum_{j=1}^{m} a_j I_j^{+} + \frac{1}{2}\sum_{j=1}^{m} a_j I_j^{-}, \qquad (5.2) \end{aligned}$$

Optical depth discretization

$$\tau_\nu^c = \int_{z_0}^{\infty} \alpha_\nu^c\,\mathrm{d}z = \int_0^{\tau_0} \mathrm{d}t^c = \int_{-\infty}^{\log\tau_0} \frac{\mathrm{d}\log t^c}{\log e} \qquad (5.3)$$

$$\tau_\nu^{\mathrm{total}} = \int_{z_0}^{\infty} (\alpha_\nu^c + \alpha_\nu^l)\,\mathrm{d}z = \int_0^{\tau_0} (1+\eta_\nu)\,\mathrm{d}t^c = \int_{-\infty}^{\log\tau_0} (1+\eta_\nu)\,\frac{\mathrm{d}\log t^c}{\log e}, \qquad (5.4)$$

General transport equation

$$\mu\,\frac{\mathrm{d}I}{\mathrm{d}\tau} = I - S. \qquad (5.5)$$

Two-level coherent scattering

$$\mu\,\frac{\mathrm{d}I}{\mathrm{d}\tau} = I - \varepsilon B - \frac{(1-\varepsilon)}{2}\int_{-1}^{+1} I\,\mathrm{d}\mu. \qquad (5.6)$$

Discrete transport equation

$$\begin{aligned} \mu\,\frac{\mathrm{d}I_\mu^{+}}{\mathrm{d}\tau} &= I_\mu^{+} - \varepsilon B - \frac{(1-\varepsilon)}{2}\left[\sum_{j=1}^{m} a_j I_j^{+} + \sum_{j=1}^{m} a_j I_j^{-}\right] \qquad (5.7) \\ (-\mu)\,\frac{\mathrm{d}I_\mu^{-}}{\mathrm{d}\tau} &= I_\mu^{-} - \varepsilon B - \frac{(1-\varepsilon)}{2}\left[\sum_{j=1}^{m} a_j I_j^{+} + \sum_{j=1}^{m} a_j I_j^{-}\right]. \qquad (5.8) \end{aligned}$$

Feautrier Method

Antisymmetric averages

$$P_\nu(\tau_\nu,\mu) \equiv \frac{1}{2}\left[I_\nu(\tau_\nu,\mu)+I_\nu(\tau_\nu,-\mu)\right] = \frac{1}{2}\left[I_j^+ + I_j^-\right] \tag{5.9}$$

$$R_\nu(\tau_\nu,\mu) \equiv \frac{1}{2}\left[I_\nu(\tau_\nu,\mu)-I_\nu(\tau_\nu,-\mu)\right] = \frac{1}{2}\left[I_j^+ - I_j^-\right], \tag{5.10}$$

Mean intensity

$$J_\nu(\tau_i) = \int_0^{+1} P_\nu(\tau_i,\mu)\,\mathrm{d}\mu \tag{5.11}$$

$$\approx \sum_{j=1}^{m} a_j P(\tau_i,\mu_j) \tag{5.12}$$

Split transport equations

$$\mu\,\frac{\mathrm{d}I^+}{\mathrm{d}\tau} = I^+ - S^+ \tag{5.13}$$

$$-\mu\,\frac{\mathrm{d}I^-}{\mathrm{d}\tau} = I^- - S^-. \tag{5.14}$$

Add and subtract; assume $S^+ = S^-$

$$\mu\,\frac{\mathrm{d}R}{\mathrm{d}\tau} = P - S \tag{5.15}$$

$$\mu\,\frac{\mathrm{d}P}{\mathrm{d}\tau} = R. \tag{5.16}$$

Feautrier transport equation

$$\mu^2\,\frac{\mathrm{d}^2P(\tau,\mu)}{\mathrm{d}\tau^2} = P(\tau,\mu) - S(\tau), \tag{5.17}$$

Numerical Feautrier solution

Outer boundary

$$\mu \left[\frac{\mathrm{d}P(\tau,\mu)}{\mathrm{d}\tau}\right]_{\tau_1} = P(\tau_1,\mu), \tag{5.18}$$

Inner boundary

$$\begin{aligned}
\mu \left[\frac{\mathrm{d}P(\tau,\mu)}{\mathrm{d}\tau}\right]_{\tau_n} &= R(\tau_n,\mu) \\
&= \frac{1}{2}\left[I(\tau_n,\mu) - I(\tau_n,-\mu)\right] \\
&= I(\tau_n,\mu) - \frac{1}{2}\left[I(\tau_n,\mu) + I(\tau_n,-\mu)\right] \\
&= I(\tau_n,\mu) - P(\tau_n,\mu) \\
&= B(\tau_n) + \mu \left[\frac{\mathrm{d}B}{\mathrm{d}\tau}\right]_{\tau_n} - P(\tau_n,\mu) \qquad (5.19)
\end{aligned}$$

Differences

$$[\Delta\tau]_{i+1/2} \approx \tau_{i+1} - \tau_i \equiv \Delta\tau_i \tag{5.20}$$

$$[\Delta\tau]_{i-1/2} \approx \tau_i - \tau_{i-1} \equiv \Delta\tau_{i-1} \tag{5.21}$$

Tridiagonal system

$$\mu^2 \left[\frac{\mathrm{d}^2P}{\mathrm{d}\tau^2}\right]_i - P_i = A_i P_{i-1} - B_i P_i + C_i P_{i+1} = -S_i \tag{5.22}$$

$$A_i = \frac{2\mu^2}{\Delta\tau_{i-1}\left(\Delta\tau_{i-1} + \Delta\tau_i\right)} \tag{5.23}$$

$$B_i = 1 + \frac{2\mu^2}{\Delta\tau_i\,\Delta\tau_{i-1}} \tag{5.24}$$

$$C_i = \frac{2\mu^2}{\Delta\tau_i\left(\Delta\tau_{i-1} + \Delta\tau_i\right)}. \tag{5.25}$$

Matrix notation

$$\mathbf{T}\,\mathbf{P} = \mathbf{S} \tag{5.26}$$

Coherent scattering

$$\mu^2 \frac{\mathrm{d}^2P(\tau,\mu)}{\mathrm{d}\tau^2} = P(\tau,\mu) - \varepsilon(\tau)B(\tau) - (1-\varepsilon(\tau))\,J(\tau) \tag{5.27}$$

$$= P(\tau,\mu) - \varepsilon(\tau)B(\tau) - (1-\varepsilon(\tau))\sum_{j=1}^{m} a_j P_j(\tau,\mu_j), \tag{5.28}$$

Lambda Iteration

Lambda operator

$$J_\nu(\tau_\nu) = \mathbf{\Lambda}_\nu[S_\nu(t)] \tag{5.29}$$

Two-level source function

$$S^l_\nu(\tau_\nu) = (1-\varepsilon_\nu(\tau_\nu))\,\mathbf{\Lambda}_\nu[S^l_\nu(t_\nu)] + \varepsilon_\nu(\tau_\nu)B_\nu(\tau_\nu). \tag{5.30}$$

Dropping indices

$$S = (1-\varepsilon)\,\mathbf{\Lambda}_\nu[S] + \varepsilon B \tag{5.31}$$

Direct solution

$$S = (1-(1-\varepsilon)\,\mathbf{\Lambda})^{-1}\,[\varepsilon B], \tag{5.32}$$

Lambda iteration

$$S^{(n+1)} = (1-\varepsilon)\,\mathbf{\Lambda}[S^{(n)}] + \varepsilon B, \tag{5.33}$$

General Lambda iteration

$$J^{(n)}_{ij} = \mathbf{\Lambda}_\nu[S^{(n)}_{ij}] \tag{5.34}$$

$$\mathbf{n}^{(n)} = f_1(J^{(n)}) \tag{5.35}$$

$$S^{(n+1)}_{ij} = f_2(\mathbf{n}^{(n)}). \tag{5.36}$$

Convergence

$$S^{(n+1)} - S^{(n)} = (1-\varepsilon)\,\mathbf{\Lambda}_\nu[S^{(n)}] + \varepsilon B - S^{(n)} \tag{5.37}$$

Large τ, small ε

$$S^{(n+1)} - S^{(n)} \approx \mathbf{\Lambda}_\nu[S^{(n)}] - S^{(n)} \approx S^{(n)} - S^{(n)} \approx 0. \tag{5.38}$$

Approximate Lambda Iteration

Operator splitting

$$\mathbf{\Lambda}_\nu = \mathbf{\Lambda}^* + (\mathbf{\Lambda}_\nu - \mathbf{\Lambda}^*) \tag{5.39}$$

Exact operation

$$J_\nu = \mathbf{\Lambda}^*_\nu[S] + (\mathbf{\Lambda}_\nu - \mathbf{\Lambda}^*_\nu)[S] \tag{5.40}$$

Iteration scheme

$$S^{(n+1)} = (1-\varepsilon)\,\mathbf{\Lambda}^*[S^{(n+1)}] + (1-\varepsilon)(\mathbf{\Lambda}_\nu - \mathbf{\Lambda}^*)[S^{(n)}] + \varepsilon B \tag{5.41}$$

$$\begin{aligned} S^{(n+1)} - (1-\varepsilon)\,\mathbf{\Lambda}^*[S^{(n+1)}] &= (1-\varepsilon)\,\mathbf{\Lambda}_\nu[S^{(n)}] + \varepsilon B - (1-\varepsilon)\,\mathbf{\Lambda}^*[S^{(n)}] \\ &= S^{\mathrm{FS}} - (1-\varepsilon)\,\mathbf{\Lambda}^*[S^{(n)}], \end{aligned} \tag{5.42}$$

Inversion

$$S^{(n+1)} = (1-(1-\varepsilon)\,\mathbf{\Lambda}^*)^{-1}\left[S^{\mathrm{FS}} - (1-\varepsilon)\,\mathbf{\Lambda}^*[S^{(n)}]\right]. \tag{5.43}$$

Convergence $\mathbf{\Lambda}_\nu$

$$S^{(n+1)} - S^{(n)} = S^{\mathrm{FS}} - S^{(n)}, \tag{5.44}$$

Convergence $\mathbf{\Lambda}^*_\nu$

$$\begin{aligned} S^{(n+1)} - (1-\varepsilon)\,\mathbf{\Lambda}^*[S^{(n+1)}] + (1-\varepsilon)\,\mathbf{\Lambda}^*[S^{(n)}] - S^{(n)} &= S^{\mathrm{FS}} - S^{(n)} \\ (1-(1-\varepsilon)\,\mathbf{\Lambda}^*)\,[S^{(n+1)}] - (1-(1-\varepsilon)\,\mathbf{\Lambda}^*)\,[S^{(n)}] &= S^{\mathrm{FS}} - S^{(n)} \end{aligned}$$

$$S^{(n+1)} - S^{(n)} = (1-(1-\varepsilon)\,\mathbf{\Lambda}^*)^{-1}\,[S^{\mathrm{FS}} - S^{(n)}]. \tag{5.45}$$

$$(1-(1-\varepsilon)\,\mathbf{\Lambda}^*)^{-1} \approx 1/\varepsilon, \tag{5.46}$$

<u>Core Saturation Operator</u>

$\tau_\nu \gg 1$

$$I_\nu(z) \approx S_\nu^{\mathrm{tot}}(z). \tag{5.47}$$

$$I_\nu(z) = S_\nu(z) \quad \text{for} \quad \tau_\nu(z) > \gamma \tag{5.48}$$

Line core operator

$$\mathbf{\Lambda}_\nu^*[S(\tau_\nu)] = S(\tau_\nu) \quad \text{or} \quad \mathbf{\Lambda}_\nu^* = 1 \quad \text{for} \quad \tau_\nu \geq \gamma \tag{5.49}$$

Thin approximation

$$J_\nu \approx \frac{1}{2} S_\nu(\tau_\nu = \gamma) \tag{5.50}$$

Line wing operator

$$\mathbf{\Lambda}_\nu^*[S(\tau_\nu)] \;\; = \;\; \frac{1}{2} S(\tau_\nu = \gamma) \quad \text{or} \quad \mathbf{\Lambda}_\nu^* = \frac{1}{2} \int \delta(\tau_\nu - \gamma)\, \mathrm{d}\tau_\nu \quad \text{for} \quad \tau_\nu < \gamma. \tag{5.51}$$

Scharmer Operator

Seek intensity operator

$$I_\nu(\tau_{\nu\mu},\mu) \equiv I^{\pm}_{\nu\mu} = \mathbf{\Lambda}^*_{\nu\mu}[S_\nu(\tau_{\nu\mu})] \approx W^{\pm}_{\nu\mu}(\tau_{\nu\mu})\, S_\nu\left(f^{\pm}_{\nu\mu}(\tau_{\nu\mu})\right), \tag{5.52}$$

Optical depth along beam

$$\mathrm{d}\tau_{\nu\mu} = (1+\eta_\nu)\,\frac{\mathrm{d}\tau^c_\nu}{|\mu|} \tag{5.53}$$

Transport equation

$$\frac{\mathrm{d}I_{\nu\mu}}{\mathrm{d}\tau_{\nu\mu}} = I_{\nu\mu} - S_\nu \tag{5.54}$$

Formal solution

$$I^+_{\nu\mu} = \mathrm{e}^{\tau_{\nu\mu}} \int_{\tau_{\nu\mu}}^{\infty} S_\nu\,\mathrm{e}^{-t_{\nu\mu}}\,\mathrm{d}t_{\nu\mu} \tag{5.55}$$

$$I^-_{\nu\mu} = \mathrm{e}^{-\tau_{\nu\mu}} \int_0^{\tau_{\nu\mu}} S_\nu\,\mathrm{e}^{t_{\nu\mu}}\,\mathrm{d}t_{\nu\mu}, \tag{5.56}$$

Linear source function

$$S_\nu = a + b\,\tau_{\nu\mu} \tag{5.57}$$

$\mu > 0$

$$I^+_{\nu\mu} = a + b + b\,\tau_{\nu\mu} \tag{5.58}$$

$\mu < 0$

$$I^-_{\nu\mu} = a - b + b\,\tau_{\nu\mu} - (a-b)\,\mathrm{e}^{-\tau_{\nu\mu}}. \tag{5.59}$$

$\mu > 0$

$$\begin{aligned} W^+_{\nu\mu} &= 1 \\ f^+_{\nu\mu} &= \tau_{\nu\mu} + 1 \end{aligned} \tag{5.60}$$

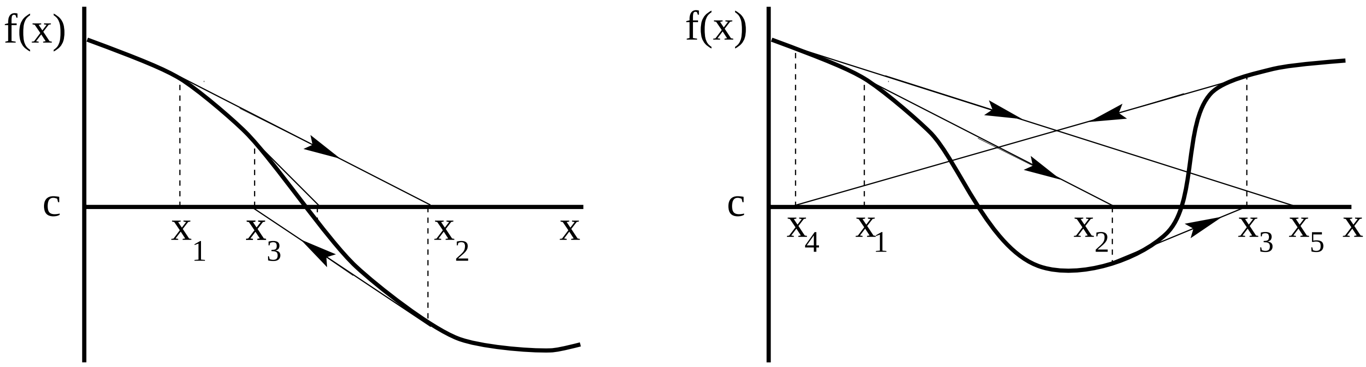
Local Eddington-Barbier

$$I^+_{\nu\mu}(\tau_{\nu\mu}) = 1 \cdot S_\nu(\tau_{\nu\mu}{+}1) = a + b\,(\tau_{\nu\mu} + 1) \tag{5.61}$$

$\mu < 0$

$$\begin{aligned} W^-_{\nu\mu} &= 1 - \mathrm{e}^{-\tau_{\nu\mu}} \\ f^-_{\nu\mu} &= \frac{\tau_{\nu\mu}}{1 - \mathrm{e}^{-\tau_{\nu\mu}}} - 1 \end{aligned} \tag{5.62}$$

Newton-Raphson Iteration



Taylor expansion

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)\,(x-x_0)^n \\ &= f(x_0) + \left[\frac{\partial f}{\partial x}\right]_{x_0} (x-x_0) + \mathcal{O}\left[(x-x_0)^2\right] \end{aligned} \tag{5.63}$$

Linearization

$$f(x_0+\Delta x) - f(x_0) \approx \left[\frac{\partial f}{\partial x}\right]_{x_0} \Delta x. \tag{5.64}$$

$$c - f(x^{(1)}) \approx \left[\frac{\partial f}{\partial x}\right]_{x^{(1)}} \Delta x^{(1)} \tag{5.65}$$

$$\Delta x^{(1)} = \frac{c - f(x^{(1)})}{[\partial f/\partial x]_{x^{(1)}}} \tag{5.66}$$

COMPLETE LINEARIZATION

Rate equations

$$n_i^{(n)} \sum_{j\neq i}^{N} P_{ij}^{(n)} - \sum_{j\neq i}^{N} n_j^{(n)} P_{ji}^{(n)} = E_i^{(n)} \tag{5.67}$$

Rates per particle

$$P_{ij} = R_{ij} + C_{ij}. \tag{5.68}$$

Perturbations

$$n_i^{(n+1)} = n_i^{(n)} + \delta n_i^{(n)} \tag{5.69}$$

$$P_{ij}^{(n+1)} = P_{ij}^{(n)} + \delta P_{ij}^{(n)} \tag{5.70}$$

$$n_i^{(n+1)} \sum_{j\neq i}^{N} P_{ij}^{(n+1)} - \sum_{j\neq i}^{N} n_j^{(n+1)} P_{ji}^{(n+1)} = 0. \tag{5.71}$$

$$\delta n_i^{(n)} \sum_{j\neq i}^{N} P_{ij}^{(n)} + n_i^{(n)} \sum_{j\neq i}^{N} \delta P_{ij}^{(n)} - \sum_{j\neq i}^{N} \delta n_j^{(n)} P_{ji}^{(n)} - \sum_{j\neq i}^{N} n_j^{(n)} \delta P_{ji}^{(n)} = -E_i^{(n)}. \tag{5.72}$$

$$\delta P_{ij}^{(n)} = B_{ij} \delta \overline{J}_{ij}^{(n)} \tag{5.73}$$

$$= \frac{B_{ij}}{2} \int_{-1}^{+1} \int_0^{\infty} \varphi(\nu - \nu_0)\, \delta I_\nu^{(n)}(\mu)\, \mathrm{d}\nu\, \mathrm{d}\mu. \tag{5.74}$$

Feautrier solution

$$\frac{\mathrm{d}^2 f_\nu J_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu), \tag{5.75}$$

$$\delta n_i = \sum_k \frac{\partial n_i}{\partial J_k} \delta J_k, \tag{5.76}$$

Approximate operator solution

$$\delta I_\nu^{(n)}(\mu) = \mathbf{\Lambda}_{\nu\mu}^{*}[\delta S_\nu^{(n)}] \tag{5.77}$$

Preconditioning

$$\mathbf{\Lambda}_{\nu\mu}^{\dagger} \equiv \mathbf{\Lambda}_{\nu\mu}^{*} - 1 \tag{5.78}$$

Chapter 6

Stokes Parameters

Electromagnetic wave

$$\begin{aligned} E_x &= A_x \cos(\omega t - \phi_x) \\ E_y &= A_y \cos(\omega t - \phi_y), \end{aligned} \tag{6.1}$$

Complete polarization

$$\begin{aligned} I_\nu &\equiv A_x^2 + A_y^2 \\ Q_\nu &\equiv A_x^2 - A_y^2 \\ U_\nu &\equiv 2A_x A_y \cos(\phi_x - \phi_y) \\ V_\nu &\equiv 2A_x A_y \sin(\phi_x - \phi_y), \end{aligned} \tag{6.2}$$

Mixed radiation

$$\begin{aligned} I_\nu &= I_\nu^{\rm unpol} + < A_x^2 + A_y^2 > \\ Q_\nu &= < A_x^2 - A_y^2 > \\ U_\nu &= < 2A_x A_y \cos(\phi_x - \phi_y) > \\ V_\nu &= < 2A_x A_y \sin(\phi_x - \phi_y) >, \end{aligned} \tag{6.3}$$

Complete polarization

$$\begin{aligned} I_\nu &= A_x^2 + A_y^2 \equiv A^2 \\ Q_\nu &= A^2 \cos 2\beta \cos 2\chi \\ U_\nu &= A^2 \cos 2\beta \sin 2\chi \\ V_\nu &= A^2 \sin 2\beta. \end{aligned} \tag{6.4}$$

Original definition

$$\begin{aligned} I_\nu &\equiv \text{total intensity} \\ Q_\nu &\equiv I_0^{\rm linear} - I_{90}^{\rm linear} \\ U_\nu &\equiv I_{+45}^{\rm linear} - I_{-45}^{\rm linear} \\ V_\nu &\equiv I_{\rm right}^{\rm circular} - I_{\rm left}^{\rm circular}. \end{aligned} \tag{6.5}$$

Chapter 7

Classical Stellar Atmospheres

Assumptions

- the atmosphere is spherically symmetric (excluding close binaries, rapid rotators, magnetic fields, spotted surfaces);
- the element mixture is homogeneous with depth;
- the atmosphere is in hydrostatic equilibrium (no large-scale motions);
- the atmosphere is time-independent (statistical equilibrium);
- the mass of the atmosphere is small compared with the total stellar mass;
- there are no sources or sinks of energy;
- energy transport takes place by radiation and convection (no heat conduction, acoustic waves, MHD waves);
- the free electrons as well as the free heavier particles obey the Maxwell distribution with local kinetic temperature T_e.

Parameters

- the stellar luminosity
- the stellar radius
- the element mixture
- the microturbulence

Two of L, M, R replaceable by

- the effective temperature $T_\mathrm{eff} = (L/4\pi\sigma R^2)^{1/4}$;
- the surface gravity $g_s = GM/R^2$.

Gas and Electron Pressure

Ideal gas

$$P_{\rm g}V = n_{\rm mole}\mathcal{R}T \tag{7.1}$$

Gas constant

$$\mathcal{R} = kN_{\rm A} = k/m_{\rm H} \tag{7.2}$$

Other versions

$$P_{\rm g} = \frac{n_{\rm mole}N_{\rm A}}{V}\frac{\mathcal{R}}{N_{\rm A}}T = N_{\rm g}kT = \frac{\rho kT}{\mu\, m_{\rm H}} = \frac{\rho\mathcal{R}T}{\mu}. \tag{7.3}$$

Electron pressure

$$P_{\rm e} = N_{\rm e}kT. \tag{7.4}$$

Ionization Balance

Hydrogen plus single metal

$$N_{\mathrm{g}} = N_{\mathrm{H}} + A_{\mathrm{M}} N_{\mathrm{H}} + f_{\mathrm{H}} N_{\mathrm{H}} + f_{\mathrm{M}} A_{\mathrm{M}} N_{\mathrm{H}} \tag{7.5}$$

Electron density

$$N_{\mathrm{e}} = f_{\mathrm{H}} N_{\mathrm{H}} + f_{\mathrm{M}} A_{\mathrm{M}} N_{\mathrm{H}}. \tag{7.6}$$

Ratio

$$\frac{N_{\mathrm{e}}}{N_{\mathrm{g}}} = \frac{f_{\mathrm{H}} + f_{\mathrm{M}} A_{\mathrm{M}}}{1 + f_{\mathrm{H}} + (1 + f_{\mathrm{M}}) A_{\mathrm{M}}} \tag{7.7}$$

High/intermediate/low temperature

$$f_{\mathrm{H}} \approx 1 \;\; \rightarrow \;\; \frac{N_{\mathrm{e}}}{N_{\mathrm{g}}} \approx \frac{1}{2} \tag{7.8}$$

$$A_{\mathrm{M}} \ll f_{\mathrm{H}} \ll 1 \;\; \rightarrow \;\; \frac{N_{\mathrm{e}}}{N_{\mathrm{g}}} \approx f_{\mathrm{H}} \tag{7.9}$$

$$f_{\mathrm{H}} \approx 0 \;\; \rightarrow \;\; \frac{N_{\mathrm{e}}}{N_{\mathrm{g}}} \approx f_{\mathrm{M}} A_{\mathrm{M}}. \tag{7.10}$$

Ionization fractions

$$\frac{1}{f_{\mathrm{II}}} = \frac{N_{\mathrm{I}}}{N_{\mathrm{II}}} + \frac{N_{\mathrm{II}}}{N_{\mathrm{II}}} + \frac{N_{\mathrm{III}}}{N_{\mathrm{II}}} \tag{7.11}$$

$$\frac{1}{f_{\mathrm{III}}} = \frac{N_{\mathrm{I}}}{N_{\mathrm{II}}} \frac{N_{\mathrm{II}}}{N_{\mathrm{III}}} + \frac{N_{\mathrm{II}}}{N_{\mathrm{III}}} + \frac{N_{\mathrm{III}}}{N_{\mathrm{III}}}. \tag{7.12}$$

Electrons per nucleus

$$E = \frac{N_{\mathrm{e}}}{N_{\mathrm{nuclei}}} = \frac{\sum_{\mathrm{z}} N_{\mathrm{z}} f_{\mathrm{II}}(z) + 2 \sum_{\mathrm{z}} N_{\mathrm{z}} f_{\mathrm{III}}(z)}{\sum_{\mathrm{z}} N_{\mathrm{z}}}, \tag{7.13}$$

Pressure ratio

$$\frac{P_{\mathrm{g}}}{P_{\mathrm{e}}} = \frac{(N_{\mathrm{ions}} + N_{\mathrm{atoms}} + N_{\mathrm{e}})\; kT}{N_{\mathrm{e}} kT} = \frac{(N_{\mathrm{nuclei}} + N_{\mathrm{e}})\, kT}{N_{\mathrm{e}} kT} = \frac{E+1}{E} \tag{7.14}$$

$$P_{\mathrm{e}} = P_{\mathrm{g}} \frac{E}{E+1}. \tag{7.15}$$

Hydrostatic Equilibrium

Pressure gradient against gravity

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -g\rho \tag{7.16}$$

Standard optical depth scale

$$\frac{\mathrm{d}P}{\mathrm{d}\tau_0} = \frac{g}{\kappa_0}. \tag{7.17}$$

Radiation pressure

$$\frac{\mathrm{d}p}{\mathrm{d}\tau_0} = \frac{4\pi}{c}\int_0^\infty \frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_0}\,\mathrm{d}\nu = \frac{4\pi}{c}\int_0^\infty \frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu}\frac{\mathrm{d}\tau_\nu}{\mathrm{d}\tau_0}\,\mathrm{d}\nu = \frac{1}{c}\int_0^\infty \mathcal{F}_\nu \frac{\kappa_\nu}{\kappa_0}\,\mathrm{d}\nu \tag{7.18}$$

Iterative integration

$$P_\mathrm{g}^{1/2}\,\frac{\mathrm{d}P_\mathrm{g}}{\mathrm{d}\tau_0} = P_\mathrm{g}^{1/2}\,\frac{g}{\kappa_0} \tag{7.19}$$

Formal solution

$$P_\mathrm{g}(\tau_0) = \left(\frac{3g}{2}\int_0^{\tau_0} \frac{P_\mathrm{g}^{1/2}(t_0)}{\kappa_0(t_0)}\,\mathrm{d}t_0\right)^{2/3}. \tag{7.20}$$

Isothermal atmosphere with constant μ

$$\frac{\mathrm{d}P_\mathrm{g}}{\mathrm{d}z} = -\frac{\mu g}{\mathcal{R}T}\,P_\mathrm{g} = -\frac{P_\mathrm{g}}{H_\mathrm{P}}, \tag{7.21}$$

$$P_\mathrm{g}(z) = P_\mathrm{g}(0)\,\mathrm{e}^{-z/H_\mathrm{P}}. \tag{7.22}$$

Plane-parallel layers

$$\frac{H_\mathrm{P}}{R_*} = \frac{\mathcal{R}T}{\mu g R_*} = \frac{\mathcal{R}T R_*}{\mu G M_*} = 4.4\times 10^{-8}\,\frac{T_\mathrm{eff}\,(R_*/R_\odot)}{\mu\,(M_*/M_\odot)} \ll 1 \tag{7.23}$$

Limb darkening

Three-parameter fit

$$\frac{I_\nu(0,\mu)}{I_\nu(0,1)} = a_\nu + b_\nu\mu + c_\nu\left(1 - \mu\,\ln(1+\frac{1}{\mu})\right) \tag{7.24}$$

$$S_\nu(\tau_\nu) = a_\nu + b_\nu\tau_\nu + c_\nu E_2(\tau_\nu) \tag{7.25}$$

Optical depth scales

$$\frac{\mathrm{d}\tau_\nu}{\mathrm{d}\tau_0} = \frac{\kappa_\nu\rho\,\mathrm{d}z}{\kappa_0\rho\,\mathrm{d}z} = \frac{\kappa_\nu}{\kappa_0} \qquad\qquad \tau_\nu(\tau_0) = \int_0^{\tau_0}\frac{\kappa_\nu}{\kappa_0}\,\mathrm{d}t_\nu \tag{7.26}$$

Radiative Equilibrium

Flux constancy

$$\nabla \cdot \mathbf{F}_{\rm tot}(\mathbf{r}) = \nabla \cdot [\mathbf{F}_{\rm rad}(\mathbf{r}) + \mathbf{F}_{\rm conv}(\mathbf{r}) + \mathbf{F}_{\rm cond}(\mathbf{r}) + \mathbf{F}_{\rm mech}(\mathbf{r})] \equiv 0, \tag{7.27}$$

Plane-parallel

$$\frac{\mathrm{d}F_{\rm tot}}{\mathrm{d}z} = 0. \tag{7.28}$$

RE

$$\mathcal{F}_{\rm rad}(z) \equiv \int_0^\infty \mathcal{F}_\nu(z)\,\mathrm{d}\nu = \mathcal{F} \tag{7.29}$$

Surface flux

$$\mathcal{F} \equiv \sigma T_{\rm eff}^4 = \frac{L_*}{4\pi R_*^2}. \tag{7.30}$$

Alternative

$$\frac{\mathrm{d}\mathcal{F}_{\rm rad}(z)}{\mathrm{d}z} = 0 \tag{7.31}$$

$$\int_0^\infty \kappa_\nu(z)\rho(z)\,J_\nu(z)\,\mathrm{d}\nu = \int_0^\infty \kappa_\nu(z)\rho(z)\,S_\nu(z)\,\mathrm{d}\nu \tag{7.32}$$

Flux divergence

$$\Phi_{\rm tot}(z) \equiv \frac{\mathrm{d}\mathcal{F}_{\rm rad}(z)}{\mathrm{d}z} = 4\pi \int_0^\infty \alpha_\nu(z)\,[S_\nu(z) - J_\nu(z)]\,\mathrm{d}\nu = 0, \tag{7.33}$$

Hubený notation

$$\Phi_{\rm tot}(z) \equiv \frac{\mathrm{d}\mathcal{F}_{\rm rad}(z)}{\mathrm{d}z} = \frac{1}{2}\int_0^\infty\int_{-1}^{+1} [j_{\nu\mu}(z) - \alpha_{\nu\mu}(z)\,I_{\nu\mu}(z)]\,\mathrm{d}\nu\,\mathrm{d}\mu = 0. \tag{7.34}$$

Net cooling rate per transition

Per line

$$\begin{aligned} \Phi_{ul} &= 4\pi\alpha^l_{\nu_0}(S^l_{\nu_0} - \overline{J}_{\nu_0}) \\ &= 4\pi j^l_{\nu_0} - 4\pi\alpha^l_{\nu_0}\overline{J}_{\nu_0} \\ &= h\nu_0\left[n_u(A_{ul} + B_{ul}\overline{J}_{\nu_0}) - n_l B_{lu}\overline{J}_{\nu_0}\right] \\ &= h\nu_0\left[n_u R_{ul} - n_l R_{lu}\right], \end{aligned} \tag{7.35}$$

Wien limit

$$\Phi_{ul} = h\nu_0\left[n_u R_{ul} - n_l R_{lu}\right] \approx 4\pi\, b_u\, \left[\alpha^l_{\nu_0}\right]_{\mathrm{LTE}} \left(B_{\nu_0} - \frac{b_l}{b_u}\overline{J}_{\nu_0}\right). \tag{7.36}$$

Per bound-free continuum

$$\Phi_{ci} = 4\pi\, n_i^{\mathrm{LTE}} b_c \int_{\nu_0}^{\infty} \sigma_{ic}(\nu)\left[B_\nu\left(1 - \mathrm{e}^{-h\nu/kT}\right) - \frac{b_i}{b_c}J_\nu\left(1 - \frac{b_c}{b_i}\,\mathrm{e}^{-h\nu/kT}\right)\right]\,\mathrm{d}\nu. \tag{7.37}$$

Wien limit

$$\Phi_{ci} = 4\pi\, n_i^{\mathrm{LTE}} b_c \int_{\nu_0}^{\infty} \sigma_{ic}(\nu)\left(B_\nu - \frac{b_i}{b_c}J_\nu\right)\,\mathrm{d}\nu. \tag{7.38}$$

Grey RE Atmosphere

Transport equation

$$\begin{aligned}\int_0^\infty \mu\frac{\mathrm{d}I_\nu(\tau_\nu,\mu)}{\mathrm{d}\tau_\nu}\,\mathrm{d}\nu &= \int_0^\infty \left[I_\nu(\tau_\nu,\mu)-S_\nu(\tau_\nu)\right]\,\mathrm{d}\nu \\ \mu\frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau} &= I(\tau,\mu)-S(\tau). \end{aligned} \tag{7.39}$$

RE condition

$$S(\tau)=J(\tau) \tag{7.40}$$

Lambda operator

$$J(\tau)=\mathbf{\Lambda}_\tau[S(t)] \tag{7.41}$$

Phi operator

$$F(\tau)=\mathbf{\Phi}_\tau[S(t)]=F. \tag{7.42}$$

Constant flux

$$S(\tau)\approx c\,(1+\frac{3}{2}\tau). \tag{7.43}$$

Formally

$$F=\mathbf{\Phi}_\tau[S(t)]=\frac{\mathrm{d}}{\mathrm{d}\tau}\chi_\tau[S(\tau)]=4\frac{\mathrm{d}K(\tau)}{\mathrm{d}\tau}, \tag{7.44}$$

Classical version

$$S(\tau)=\frac{3}{4}(\tau+q(\tau))\,F \tag{7.45}$$

Hopf function

$$\tau+q(\tau)=\mathbf{\Lambda}_\tau[\tau+q(\tau)]. \tag{7.46}$$

Milne-Eddington approximation

$$S(\tau)\approx\frac{3}{4}(\tau+\frac{2}{3})\,F=(\frac{3}{4}\tau+\frac{1}{2})\,F=\frac{1}{2}(1+\frac{3}{2}\tau)\,F \tag{7.47}$$

Grey RE temperature

$$T(\tau)\approx T_{\mathrm{eff}}\left(\frac{3}{4}\tau+\frac{1}{2}\right)^{1/4} \tag{7.48}$$

Monochromatic source function

$$S_\nu(\tau)=B_\nu\left[T(\tau)\right] \tag{7.49}$$

Grey RE scattering

Thomson scattering

$$S_\nu = (1-\varepsilon_\nu) J_\nu + \varepsilon_\nu B_\nu \tag{7.50}$$

RE condition

$$\begin{aligned}
\int_0^\infty \kappa_\nu \, \rho \, J_\nu \, \mathrm{d}\nu &= \int_0^\infty \kappa_\nu \, \rho \, S_\nu \, \mathrm{d}\nu \\
\kappa \, \rho \, J &= \kappa \, \rho \, [(1-\varepsilon) J + \varepsilon B] \\
\varepsilon J &= \varepsilon B \\
J &= B \, = \, S,
\end{aligned} \tag{7.51}$$

Grey RE limb darkening

$$\frac{I(0,\mu)}{I(0,1)} = \frac{3}{5} \, (\mu + \frac{2}{3}). \tag{7.52}$$

Schwarzschild (1906)

		Limb darkening $I(0,\mu)/I(0,1)$				
$r/R_\odot$	μ	Observed	Radiative equilibrium		Convective equilibrium	
0.00	1.00	1.00	1.00	1.00	1.00	1.00
0.20	0.98	0.99	0.99	0.99	0.98	0.97
0.40	0.92	0.97	0.95	0.95	0.92	0.87
0.60	0.80	0.92	0.87	0.88	0.80	0.70
0.80	0.60	0.81	0.73	0.76	0.60	0.44
0.90	0.44	0.70	0.63	0.66	0.44	0.27
0.98	0.20	0.49	0.47	0.52	0.20	0.08
1.00	0.00	≈ 0.40	0.33	0.40	0.00	0.00

Grey Extinction and Mean extinction

Monochromatic transport equation

$$4\frac{\mathrm{d}K_\nu(z)}{\mathrm{d}\tau_\nu} = 4\frac{\mathrm{d}K_\nu(z)}{-\kappa_\nu(z)\rho(z)\,\mathrm{d}z} = F_\nu(z) \tag{7.53}$$

Mean extinction

$$\int_0^\infty 4\frac{\mathrm{d}K_\nu(z)}{-\kappa_\nu(z)\rho(z)\,\mathrm{d}z}\,\mathrm{d}\nu \equiv \frac{1}{\overline{\kappa}(z)}\int_0^\infty 4\frac{\mathrm{d}K_\nu(z)}{-\rho(z)\,\mathrm{d}z}\,\mathrm{d}\nu. \tag{7.54}$$

$$\int_0^\infty 4\frac{\mathrm{d}K_\nu(z)}{-\kappa_\nu(z)\rho(z)\,\mathrm{d}z}\,\mathrm{d}\nu = \int_0^\infty F_\nu(z)\,\mathrm{d}\nu = F(z) \tag{7.55}$$

Flux weighted mean

$$\overline{\kappa}(z) \equiv \frac{\int_0^\infty \kappa_\nu(z)\,F_\nu(z)\,\mathrm{d}\nu}{\int_0^\infty F_\nu(z)\,\mathrm{d}\nu} = \int_0^\infty \kappa_\nu(z)\,\frac{F_\nu(z)}{F(z)}\,\mathrm{d}\nu. \tag{7.56}$$

$$\frac{1}{\overline{\kappa}(z)}\int_0^\infty 4\frac{\mathrm{d}K_\nu(z)}{-\rho(z)\,\mathrm{d}z}\,\mathrm{d}\nu = \frac{1}{\overline{\kappa}(z)}\,4\frac{\mathrm{d}K(z)}{-\rho(z)\,\mathrm{d}z}, \tag{7.57}$$

$$\overline{\frac{1}{\kappa(z)}} \equiv \frac{\int_0^\infty [1/\kappa_\nu(z)]\,(\mathrm{d}K_\nu(z)/\mathrm{d}z)\,\mathrm{d}\nu}{\int_0^\infty (\mathrm{d}K_\nu(z)/\mathrm{d}z)\,\mathrm{d}\nu} = \int_0^\infty \frac{1}{\kappa_\nu(z)}\,\frac{\mathrm{d}K_\nu(z)/\mathrm{d}z}{\mathrm{d}K(z)/\mathrm{d}z}\,\mathrm{d}\nu. \tag{7.58}$$

Eddington approximation plus LTE

$$K_\nu \approx \frac{1}{3}J_\nu \approx \frac{1}{3}B_\nu \tag{7.59}$$

Rosseland extinction

$$\frac{1}{\overline{\kappa}} \approx \int_0^\infty \frac{1}{\kappa_\nu}\,\frac{\mathrm{d}B_\nu/\mathrm{d}T}{\mathrm{d}B/\mathrm{d}T}\,\mathrm{d}\nu \equiv \frac{1}{\kappa_\mathrm{R}}. \tag{7.60}$$

Stellar interior

$$T(\tau_\mathrm{R}) = T_\mathrm{eff}\left[\frac{3}{4}\tau_\mathrm{R} + \frac{3}{4}q(\tau_\mathrm{R})\right]^{1/4} \tag{7.61}$$

Rosseland optical depth

$$\mathrm{d}\tau_\mathrm{R} = -\kappa_\mathrm{R}\rho\,\mathrm{d}z. \tag{7.62}$$

Radiation

$$J(\tau_\mathrm{R}) = S(\tau_\mathrm{R}) = B(\tau_\mathrm{R}) = \frac{\sigma}{\pi}T^4(\tau_\mathrm{R}) = \frac{3}{4}\left[\tau_\mathrm{R} + q(\tau_\mathrm{R})\right]\,F, \tag{7.63}$$

LINE BLANKETING

Backwarming

$$\frac{\int_0^\infty F'_\nu \,\mathrm{d}\nu}{\int_0^\infty F_\nu \,\mathrm{d}\nu} = \frac{F'}{F} = \frac{(\sigma/\pi)\, {T'_{\mathrm{eff}}}^4}{(\sigma/\pi)\, {T_{\mathrm{eff}}}^4} = 1 - f \tag{7.64}$$

$$T_{\mathrm{eff}} = (1-f)^{-1/4}\, T'_{\mathrm{eff}} \approx (1 + f/4)\, T'_{\mathrm{eff}}. \tag{7.65}$$

Cooling rate per line

$$\Phi_{ul}(z) = 4\pi\, \alpha^l_{\nu_0}(z) \left[S^l_{\nu_0}(z) - \overline{J}_{\nu_0}(z)\right]; \tag{7.66}$$

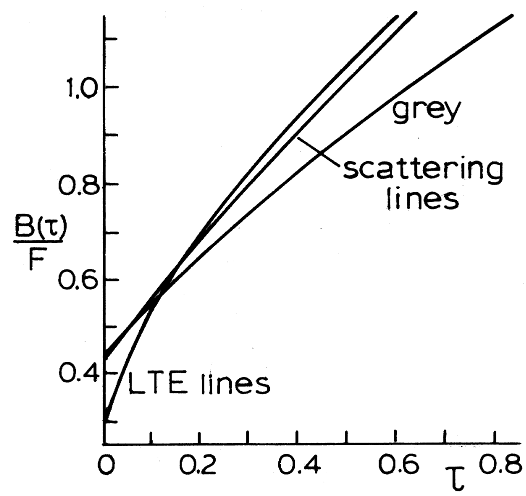
LTE RE condition

$$\int_0^\infty \kappa_\nu(0)\, J_\nu(0)\, \mathrm{d}\nu = \int_0^\infty \kappa_\nu(0)\, B_\nu(0)\, \mathrm{d}\nu \tag{7.67}$$

Scattering lines

$$S^l_{\nu_0}(0) - \overline{J}_{\nu_0}(0) \approx \frac{\varepsilon_{\nu_0}}{1 + \sqrt{\varepsilon_{\nu_0}}}\, B_{\nu_0} \approx \varepsilon_{\nu_0} B_{\nu_0}. \tag{7.68}$$

Münch (1946)



Chapter 8

Weighting function

$$G \equiv \frac{\mathrm{d}B_\nu/\mathrm{d}T}{\mathrm{d}B/\mathrm{d}T} \tag{8.1}$$

H I/H$^-$ ratio for $\log P_\mathrm{e} = 1.3$, $T_\mathrm{e} = 6000$

$$\log \frac{N(\mathrm{H\,I})}{N(\mathrm{H}^-)} = -0.1761 - \log P_\mathrm{e} + \log \frac{U(\mathrm{H\,I})}{U(\mathrm{H}^-)} + 2.5 \log T_\mathrm{e} - \theta\,\chi = 7.64 \tag{8.2}$$

Boltzmann factor

$$\mathrm{e}^{-h\nu/kT} = 10^{-1.6021\times10^{-12}\ \log(\mathrm{e})\ (\chi/kT)} = 10^{-(5040/T)\,\chi} = 10^{-\theta\,\chi} \tag{8.3}$$

Paschen continuum

$$\frac{n_3}{N(\mathrm{H\,I})} \approx \frac{n_3}{n_1} = \frac{g_3}{g_1}\,\mathrm{e}^{-h\nu/kT} = \frac{18}{2}\,10^{-\theta\,\chi} = 9 \times 10^{-10.1556} = 6.2 \times 10^{-10} \tag{8.4}$$

Rydberg series limits

$$\lambda_n \sim 91.16\,\frac{n^2}{Z^2} \tag{8.5}$$

n		1	2	3	4	5	6	7	8	...
H I	n^2/Z^2	1	4	9	16	25	36	49	64	...
He II	n^2/Z^2	1/4	1	9/4	4	25/4	9	49/4	16	...

The Balmer Discontinuity

Balmer jump in F and G stars

$$\frac{\kappa(\lambda > 364.7)}{\kappa(\lambda < 364.7)} = \frac{\sigma_\lambda(\mathrm{H}^-)\, N(\mathrm{H}^-)}{\sigma_\lambda(\mathrm{H}^-)\, N(\mathrm{H}^-) + \sigma_\lambda^{\mathrm{B}}\, N_{\mathrm{H}}(n{=}2)} < 1, \tag{8.6}$$

Balmer jump in F stars

$$\frac{\kappa(\lambda > 364.7)}{\kappa(\lambda < 364.7)} \sim \frac{\sigma_\lambda(\mathrm{H}^-) N_{\mathrm{H}}(n{=}1)}{\sigma_\lambda^{\mathrm{B}} N_{\mathrm{H}}(n{=}2)}\, N_{\mathrm{e}}\, T_{\mathrm{e}}^{-3/2}\, \mathrm{e}^{h\nu/kT} \sim \frac{\sigma_\lambda(\mathrm{H}^-)}{\sigma_\lambda^{\mathrm{B}}}\, N_{\mathrm{e}}\, T_{\mathrm{e}}^{-3/2}\, \mathrm{e}^{2h\nu/kT}. \tag{8.7}$$

Balmer jump in hotter stars

$$\frac{\kappa(\lambda > 364.7)}{\kappa(\lambda < 364.7)} \sim \frac{\sigma_\lambda^{\mathrm{P}}\, N_{\mathrm{H}}(n=3)}{\sigma_\lambda^{\mathrm{B}}\, N_{\mathrm{H}}(n=2)} \sim \mathrm{e}^{-h\nu/kT_{\mathrm{e}}}. \tag{8.8}$$

Chapter 9

Abundance Determination

Abundance of element E

$$A_{\mathrm{E}} \equiv \frac{N_{\mathrm{E}}}{N_{\mathrm{H}}}, \tag{9.1}$$

H = 12 scale

$$A_{12}(\mathrm{E}) \equiv \log N_{\mathrm{E}} - \log N_{\mathrm{H}} + 12. \tag{9.2}$$

Relative abundance

$$[X] \equiv \log X_{\mathrm{star}} - \log X_{\mathrm{Sun}}. \tag{9.3}$$

Metallicity

$$[\mathrm{Fe/H}] = \log(N_{\mathrm{Fe}}/N_{\mathrm{H}})_{\mathrm{star}} - \log(N_{\mathrm{Fe}}/N_{\mathrm{H}})_{\mathrm{Sun}} \tag{9.4}$$

Population

$$n_l = b_l\, n_l^{\mathrm{LTE}} = b_l\, \frac{n_l^{\mathrm{LTE}}}{N_{\mathrm{E}}}\, N_{\mathrm{H}} A_{\mathrm{E}}, \tag{9.5}$$

Line extinction coefficient

$$\alpha_\lambda^l = \frac{\sqrt{\pi} e^2}{m_{\mathrm{e}} c}\, \frac{\lambda^2}{c} b_l\, \frac{n_l^{\mathrm{LTE}}}{N_{\mathrm{E}}}\, N_{\mathrm{H}}\, A_{\mathrm{E}}\, f_{lu}\, \frac{H(a,v)}{\Delta\lambda_{\mathrm{D}}} \left[1 - \frac{b_u}{b_l}\, \mathrm{e}^{-hc/\lambda kT}\right] \tag{9.6}$$

Equivalent width

$$W_\lambda = \int_{\mathrm{line}} \frac{I_c - I_\lambda^l}{I_c}\, \mathrm{d}\lambda \tag{9.7}$$

$$W_\lambda = \int_{\mathrm{line}} \frac{\mathcal{F}_c - \mathcal{F}_\lambda^l}{\mathcal{F}_c}\, \mathrm{d}\lambda. \tag{9.8}$$

Curve of Growth

Schuster-Schwarzschild atmosphere

$$I_\lambda = B_\lambda(T_\mathrm{R})\,\mathrm{e}^{-\tau_\lambda} + B_\lambda(T_\mathrm{L})(1-\mathrm{e}^{-\tau_\lambda}) \tag{9.9}$$

Optical thickness of reversing layer

$$\tau_\lambda = \sigma_\lambda\,N_\mathrm{i} = \frac{\sqrt{\pi}\mathrm{e}^2}{m_\mathrm{e}c}\frac{\lambda_0^2}{c}\frac{f}{\Delta\lambda_\mathrm{D}}\,N_\mathrm{i}\,H(a,v) \approx \tau_{\lambda_0}\,H(a,v) \tag{9.10}$$

Line depression

$$D_\lambda \equiv \frac{I_\mathrm{c}-I_\lambda}{I_\mathrm{c}} = \frac{B_\lambda(T_\mathrm{R})-B_\lambda(T_\mathrm{L})}{B_\lambda(T_\mathrm{R})}\,(1-\mathrm{e}^{-\tau_\lambda}) = D_\mathrm{max}(1-\mathrm{e}^{-\tau_\lambda}) \tag{9.11}$$

Line depth

$$D_\mathrm{max} \equiv \frac{B_\lambda(T_\mathrm{R})-B_\lambda(T_\mathrm{L})}{B_\lambda(T_\mathrm{R})} \tag{9.12}$$

Equivalent width

$$W_\lambda = D_\mathrm{max}\int_\mathrm{line}(1-\mathrm{e}^{-\tau_\lambda})\,\mathrm{d}\lambda. \tag{9.13}$$

Weak lines

$$D_\lambda \approx D_\mathrm{max}\,\tau_{\lambda_0}\,\mathrm{e}^{-(\Delta\lambda/\Delta\lambda_\mathrm{D})^2} \tag{9.14}$$

$$W_\lambda \approx D_\mathrm{max}\tau_{\lambda_0}\sqrt{\pi}\,\Delta\lambda_\mathrm{D} = \frac{\pi\mathrm{e}^2}{m_e c}\frac{\lambda_0^2}{c}\,f\,D_\mathrm{max}\,N_\mathrm{i}. \tag{9.15}$$

Saturated lines

$$W_\lambda \approx Q\,D_\mathrm{max}\,\Delta\lambda_\mathrm{D} \tag{9.16}$$

Strong lines

$$\tau_\lambda = \tau_{\lambda_0}\frac{a}{\sqrt{\pi}v^2} = \tau_{\lambda_0}\frac{a}{\sqrt{\pi}}\frac{\Delta\lambda_\mathrm{D}^2}{\Delta\lambda^2} \tag{9.17}$$

$$\begin{aligned} W_\lambda &= D_\mathrm{max}\int_\mathrm{line}(1-\mathrm{e}^{-\tau_\lambda})\,\mathrm{d}\lambda \\ &= D_\mathrm{max}\Delta\lambda_\mathrm{D}\sqrt{\tau_{\lambda_0}(a/\sqrt{\pi})}\int_\mathrm{line}(1-\mathrm{e}^{-1/u^2})\,\mathrm{d}u \\ &\sim D_\mathrm{max}\Delta\lambda_\mathrm{D}\sqrt{\tau_{\lambda_0}\,a} \end{aligned} \tag{9.18}$$

Curve of Growth

Milne-Eddington atmosphere

$$B_\lambda(\tau_\lambda) = B_0 + \frac{b_c}{1+\eta_\lambda}\tau_\lambda \tag{9.19}$$

$$F_\lambda(0) = B_0 + \frac{b_c}{1+\eta_\lambda}\,\frac{2}{3} \tag{9.20}$$

$$\begin{aligned} D_\lambda &\equiv \frac{F_c(0) - F_\lambda(0)}{F_c(0)} \\ &= \frac{(2/3)\,b_c\,\eta_\lambda/(1+\eta_\lambda)}{B_0 + (2/3)b_c} \\ &= D_{\max}\,\frac{\eta_\lambda}{1+\eta_\lambda} \end{aligned} \tag{9.21}$$

Equivalent width

$$W_\lambda = \int_{\mathrm{line}} D_\lambda\,\mathrm{d}\lambda = D_{\max}\,\Delta\lambda_D \int_{\mathrm{line}} \frac{\eta_v}{1+\eta_v}\,\mathrm{d}v$$

$$\frac{W_\lambda}{D_{\max}\Delta\lambda_D} = \int_{\mathrm{line}} \frac{\eta_0 H(a,v)}{1+\eta_0 H(a,v)}\,\mathrm{d}v \tag{9.22}$$

$$\frac{W_\lambda}{D_{\max}\Delta\lambda_D} = \sqrt{\pi}\,\eta_0 \quad \text{for } \eta_0 \ll 1 \tag{9.23}$$

$$\frac{W_\lambda}{D_{\max}\Delta\lambda_D} = 2 - 4 \quad \text{for } \eta_0 > 1 \tag{9.24}$$

$$\frac{W_\lambda}{D_{\max}\Delta\lambda_D} = \sqrt{\pi^{3/2}\,a\,\eta_0} \quad \text{for } \eta_0 \gg 1. \tag{9.25}$$

Empirical Curve of Growth

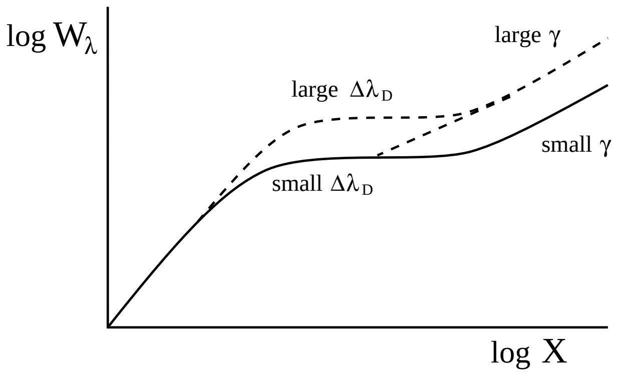
Abscissa

$$\log X = \log C + \log(gf\lambda_0) - \chi\,5040/T_{\rm exc} \tag{9.26}$$

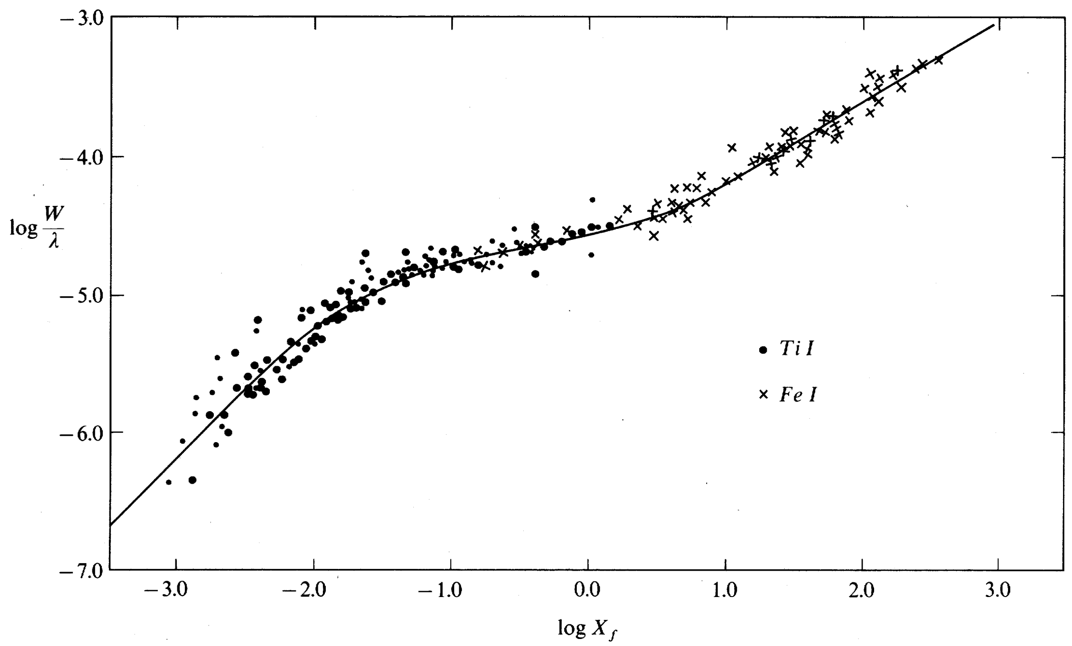
Boltzmann

$$N_{\rm i} \sim g_{\rm i}\,{\rm e}^{-E_{\rm i}/kT} = g_{\rm i}\,10^{-\chi_{\rm i}\,5040/T_{\rm exc}} \tag{9.27}$$

Sensitivities



Wright (1948)



Chapter 10

Radiative Transfer Rap

specific intensity	$I_\nu(\vec{r},\vec{l},t)$ erg cm^{-2} s^{-1} Hz^{-1} ster^{-1}
emission coefficient	j_ν erg cm^{-3} s^{-1} Hz^{-1} ster^{-1}
extinction coefficient	α_ν cm^{-1} σ_ν cm^2 part^{-1} κ_ν cm^2 g^{-1}
source function	$S_\nu = \Sigma\, j_\nu / \Sigma\, \alpha_\nu$
radial optical depth	$\tau_\nu(z_0) = \int_{z_0}^{\infty} \alpha_\nu \,\mathrm{d}z$
plane-parallel transport	$\mu \,\mathrm{d}I_\nu/\mathrm{d}\tau_\nu = I_\nu - S_\nu$
incoming rays	$I_\nu^-(\tau_\nu,\mu) = -\int_0^{\tau_\nu} S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/\mu$
outgoing rays	$I_\nu^+(\tau_\nu,\mu) = +\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/\mu$
emergent intensity	$I_\nu^+(0,\mu) = \int_0^{\infty} S_\nu(t_\nu)\,\mathrm{e}^{-t_\nu/\mu}\,\mathrm{d}t_\nu/\mu$
Eddington-Barbier	$I_\nu^+(0,\mu) \approx S_\nu(\tau_\nu=\mu)$
	$\mathcal{F}_\nu^+(0) \approx \pi S_\nu(\tau_\nu = 2/3)$
mean intensity	$J_\nu(z) = \frac{1}{2}\int_{-1}^{+1} I_\nu\,\mathrm{d}\mu$
flux	$H_\nu(z) = \frac{1}{2}\int_{-1}^{+1} \mu\, I_\nu\,\mathrm{d}\mu = \mathcal{F}_\nu/4\pi = F_\nu/4$
mean mean intensity	$\overline{J}_{\nu_0}^{\varphi} = \frac{1}{2}\int_0^{\infty}\int_{-1}^{+1} I_\nu\,\varphi(\nu-\nu_0)\,\mathrm{d}\mu\,\mathrm{d}\nu$
Schwarzschild equation	$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^{\infty} S_\nu(t_\nu)\,E_1(\lvert t_\nu-\tau_\nu\rvert)\,\mathrm{d}t_\nu$
lambda operator	$\mathbf{\Lambda}_\nu[S_\nu(t_\nu)] = J_\nu(\tau_\nu)$
photon destruction	$\varepsilon_\nu = \alpha_\nu^{\mathrm{a}}/(\alpha_\nu^{\mathrm{a}}+\alpha_\nu^{\mathrm{s}})$
coherent scattering	$S_\nu^l = (1-\varepsilon_\nu)\,J_\nu + \varepsilon_\nu B_\nu$
two-level atoms	$\varepsilon_{\nu_0} = C_{21}/(C_{21}+A_{21}+B_{21}B_\nu)$
complete redistribution	$S_{\nu_0}^l = (1-\varepsilon_{\nu_0})\,\overline{J}_{\nu_0}^{\varphi} + \varepsilon_{\nu_0}B_{\nu_0}$
isothermal atmosphere	$S_\nu(0) = \sqrt{\varepsilon_\nu}\,B_\nu$